QFT methods in gravity

In this lecture we switch gears and discuss about gravity from quantum field theory perspective.

- It is usually said that quantum mechanics and general relativity are not compatible with each other. This is true if we want to look for a quantum theory of gravity that makes sense at all energies. On the other hand if we are interested in processes only at low energies ($\ll M_{Pl} \sim \frac{1}{\sqrt{G}}$) Then general relativity is perfectly fine a quantum effective field theory (EFT).

More on this line : [gr-qc/9405057] J. Donoghue
EPFL lectures 1702.00319

For instance you can calculate quantum corrections to the Newtonian potentials. (as we do in QED for Coulomb).

- It is also possible to use QFT techniques to calculate classical observables: ones that we normally do using GR equations: 1) It is always nice to have different approaches for a same problem
  2) Perturbative techniques are very well-developed in QFT language. So this might be useful.

So I will briefly talk about this.

- Finally, I will try to convince you that putting quantum mechanics & relativity ideas is very constraining and leads to GR at low energies for massless spin 2 particles

1 Propagator of graviton

We have already wrote the quadratic action for linear metric perturbation:

$$S = \frac{-1}{64\pi G}\int d^4x \left[(\partial_\mu h_{\alpha\beta})^2 - (\partial_\mu h)^2 + 2\,\partial_\alpha h^{\alpha\nu}\partial_\nu h - 2(\partial_\alpha h^{\alpha\mu})^2 \right]$$

We can follow exactly same routes we learn in QFT courses to quantize this action: creation / annihilation operators etc. : the associated particle is called graviton. (simlar to photon or gluon)

Notice that we usually work with a kinetic term that is of the form : $\int d^4x \, -\frac{1}{2}(\partial_\mu \phi)^2 + \cdots$

this is known as canonical normalization. Canonically normalized field in this case

$$\boxed{h_c^{\mu\nu} \equiv \frac{1}{\sqrt{32\pi G}} h^{\mu\nu}} \rightsquigarrow h = k h_c$$

$\sqrt{32\pi G} \to k$

We define the propagator as the inverse of the operator appearing in the quadratic action (with an i) :

$$\text{ex: } S = \int -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 = \frac{1}{2}\int \varphi \underbrace{(\Box - m^2)}_{-k^2 - m^2}\varphi$$

$$\langle T\varphi\varphi\rangle \equiv \frac{1}{i}\Delta(k) = \frac{1}{i}\frac{1}{k^2+m^2-i\epsilon}$$

Feynman priscription different from retarded priscription!

We must do the same for the above action : there is a subtlety due to gauge invariance : the operator is not invertible [same happens for QED due to gauge inv.]

We add <u>gauge fixing terms</u> to cure this:

$$S_{gf} = -\int d^4x \, (\partial_\nu h_c^{\nu\mu})^2$$

canonical

Then: $$S + S_{gf} = -\frac{1}{2}\int d^4x \left[(\partial_\mu h_c^{\alpha\beta})^2 - \frac{1}{2}(\partial_\gamma h_c)^2 \right]$$

$$= \frac{1}{2}\int d^4x \; h_{c\,\alpha\beta} \left[\frac{1}{2}\eta^{\alpha\delta}\eta^{\beta\lambda} + \frac{1}{2}\eta^{\alpha\lambda}\eta^{\beta\delta} - \frac{1}{2}\eta^{\alpha\beta}\eta^{\delta\lambda} \right] \Box h_{c\,\delta\lambda}$$

$$\langle T h_c^{\mu\nu} h_c^{\alpha\beta} \rangle \equiv \frac{1}{i}\Delta^{\mu\nu,\alpha\beta}_{(k)} = \boxed{\frac{1}{i}\,\frac{\frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}\right)}{k^2 - i\epsilon}}$$

no mass term.

- We can also look at <u>interactions</u>:

From def. of energy-mom. tensor we have

$$S_{int} = \frac{1}{2}\int d^4x \; T_{\mu\nu} h^{\mu\nu} = \boxed{\frac{\kappa}{2}\int d^4x \; T_{\mu\nu} h_c^{\mu\nu}}$$

There are <u>self-interaction terms</u>: either by looking at cubic terms in the action, or by remembering the form of EMT of GW: $t_{\mu\nu} \sim (\partial h)^2$

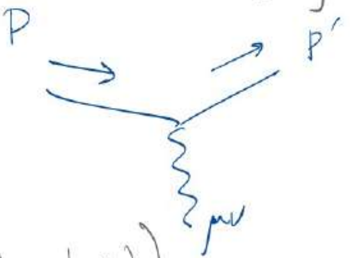
$$S_{self-int} \sim \int d^4x \; h(\partial h)^2 \sim \underline{\kappa^3 \int h_c(\partial h_c)^2} + \dots$$

2 Scattering amplitude

We can now look at scattering amplitude of two objects with mass m_1 and m_2. We model them as two scalar field with the same mass: we know that scalar field $\rightsquigarrow$ spinless point particle: so as long as this should give consistent results for low energy scattering where we are not sensitive to the inner structure of the objects.

$$S = -\frac{1}{2}\int d^4x \left[(\partial\varphi)^2 + m^2\varphi^2 \right]$$

$$T^{\mu\nu} = \partial^\mu\varphi\,\partial^\nu\varphi + \eta^{\mu\nu}\mathcal{L} \qquad S_{int} = \frac{\kappa}{2}\int T_{\mu\nu}\, h_c^{\mu\nu}$$



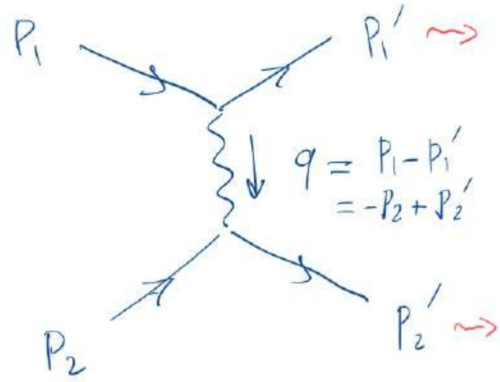
The vertex in momentum space:

$$i\kappa\left(ip^\mu(-ip'^\nu) + \eta^{\mu\nu}\left(-\frac{1}{2}(ip_\alpha)(-ip'^\alpha) - \frac{1}{2}m^2 \right) \right)$$

must be symm.

$$= \boxed{\frac{i\kappa}{2}\left[p^\mu p'^\nu + p^\nu p'^\mu - \eta^{\mu\nu}(p\cdot p' + m^2) \right]}$$

Then the amplitude for scattering will be



$\rightsquigarrow \; p_1^2 = p_1'^2 = -m_1^2$

$\rightsquigarrow \; p_2^2 = p_2'^2 = -m_2^2$

$$iM = \frac{i\kappa}{2}\left(\cdots\right)^{\mu\nu} \frac{1}{i}\, \frac{P_{\mu\nu|\alpha\beta}}{q^2}\, \frac{i\kappa}{2}\left(\cdots\right)^{\alpha\beta}$$

$$M = \frac{\kappa^2}{8q^2}\left[p_1^\mu p_1'^\nu + p_1^\nu p_1'^\mu - \eta^{\mu\nu}(p_1\cdot p_1' + m_1^2)\right]\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right]\left[\begin{matrix} 1 \to 2 \\ \mu\nu \to \alpha\beta \end{matrix}\right]$$

after some lines of algebra this simplifies as

$$\boxed{M = 16\pi G\, \frac{2(p_1\cdot p_2)^2 + (p_1\cdot p_2)q^2 - m_1^2 m_2^2}{q^2}}$$

or in terms of Mandelstam variables : $s = -(p_1+p_2)^2$, $t = -(p_1 - p_1')^2$

$$M = \frac{-8\pi G}{t}\left[s(s+t) - (m_1^2+m_2^2)(2s+t) + (m_1^4+m_2^4)\right]$$

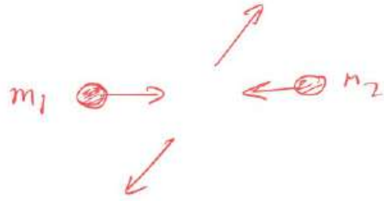
Newtonian potential

Consider two non-relativistic particles, i.e. momentum is dominated by mass :

CM frame

$$P_1^\mu \simeq (m_1, \vec{p}) \qquad P_2^\mu \simeq (m_2, -\vec{p})$$

$$P_1'^\mu \simeq (m_1, \vec{p}+\vec{q}) \qquad P_2'^\mu \simeq (m_2, -\vec{p}-\vec{q})$$



$\vec{p}, \vec{q} \ll m_1, m_2$

non-relativistic particles

with momentum transfer $q^\mu = (0, -\vec{q})$.

Then the amplitude will be :

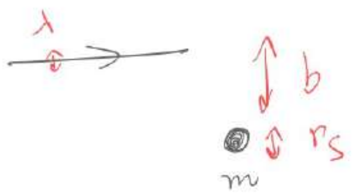
$$\mathcal{M} \simeq \frac{16\pi G m_1^2 m_2^2}{\vec{q}^{\,2}} \xrightarrow[\frac{1}{2m_1} \times \frac{1}{2m_2}]{\text{NR normalization of 1-particle states}} \mathcal{M}^{NR} \simeq \frac{4\pi G m_1 m_2}{\vec{q}^{\,2}}$$

In the limit that one of them is much more massive : $m_1 \gg m_2$, the other moves in the potential of the other one. Then we remember from QM course that, in the Born app., the amplitude will be Fourier transform of the potential.

$$V(\vec{x}) = -\int \frac{d^3\vec{q}}{(2\pi)^3} \mathcal{M}^{NR}(\vec{q})\, e^{i\vec{q}\cdot\vec{x}} = \boxed{-\frac{G m_1 m_2}{r}}$$

Light deflection & Time delay

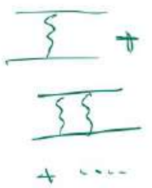
Another regime that we can look at is that one of the particles is relativistic ($m_2 = 0$), and its wavelength is much smaller than the Schwarzchild radius of the other object ($2Gm$). In that case we can treat the other particle in the geometric optic regime: like the way we solve the problem in GR: geodesic in Schwarzchild metric: A notion which is useful to talk about here is the concept of impact parameter: b



$$\lambda \ll r_s \ll b$$

The corresponding regime from QM scattering is known as eikonal scattering: you may have seen this in QM course (see e.g. Sakurai): this basically WKB approximation for the wave-function: $\psi \sim e^{iS}$ and solve for ray trajectories.

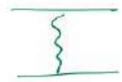
It is possible to show that in this limit the amplitude can be written in the following form:



$$i\mathcal{M}_{eik}(\underbrace{s}_{\text{CM energy}},\ \underbrace{\vec{q}}_{\text{mom trans}}) = 2(s-m^2)\int \underbrace{d^2\vec{b}}_{\text{is the analog of impact parameter}}\ e^{-i\vec{q}\cdot\vec{b}}\left[e^{i\delta(s,b)}-1\right]$$

eikonal phase shift.

And the eikonal phase shift has a simple form in terms of the tree level amplitude we derived above:



$$\delta(E,\vec{q}) = \frac{1}{2(s-m^2)}\int \frac{d^2\vec{q}}{(2\pi)^2}\, e^{i\vec{q}\cdot\vec{b}}\,\mathcal{M}_{tree}(E,\vec{q})$$

Notice that the integral is $d^2\vec{q}$ since in this limit, $\underline{\vec{q} \ll \vec{p}}$ so it only changes its direction not amplitude and therefore it is $\underline{\text{orthogonal}}$ to $\vec{p}$.

The kinematics is as follows: $p_1^\mu = (m, \vec{0})$ $\quad p_1'^\mu \simeq (m, -\vec{q})$

$p_2^\mu = (p, \vec{p})$ $\quad p_2'^\mu \simeq (p, \vec{p}+\vec{q})$

$p \equiv \omega$ energy of massless particle.

The tree level amplitude can be seen to be

$$\mathcal{M}_{tree} \simeq \frac{32\pi G\, m^2 \omega^2}{q^2}$$

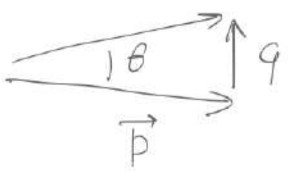
Then the eikonal phase shift is given by

$$\delta = 2Gm\omega\, \Gamma\!\left(\frac{d-4}{2}\right) \frac{1}{b^{\frac{d-4}{2}}} = 4Gm\omega\left(\frac{-1}{2\varepsilon} - \frac{\gamma}{2} - \log b\right) + \cdots$$

the integral over q
is divergent, we need to
regularize it using dim-reg: $d = 4 - 2\varepsilon$

- The deflection angle is related to $\delta(b)$ by saddle point app.

$$e^{-i\vec{q}\cdot\vec{b} + i\delta} \longrightarrow \vec{q}_* = \frac{\partial \delta}{\partial \vec{b}} \Longrightarrow \boxed{\theta \simeq \frac{1}{\omega}\frac{\partial \delta}{\partial b}}$$



$$q \simeq p\,\theta$$

$$\boxed{\theta = -\frac{4Gm}{b}}$$

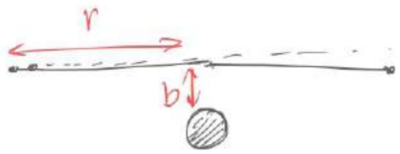
exactly what we get from Einstein gravity

- For the time delay we notice that the amplitude is "like" time evolution operator in the presence of interaction "minus" free propagation: $\mathcal{M}_{eik} \sim e^{i\delta}$ then

$$\delta = \omega \underbrace{\Delta t}$$

delay in propagation of the wave compared to free propagation.

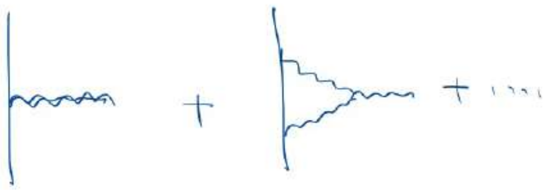
therefore $\boxed{\Delta t = 4Gm \log\left(b_0/b\right)}$ which is the expression for Shapiro time delay in GR. (b_0 is some IR scale :



$$\Delta t_{GR} = 4Gm \log\left(\frac{2r}{b}\right)$$

Schwarzschild solution

Using similar methods it is possible to obtain the Schwarzschild solution, by summing over diagrams



(3) Massive has spin, Massless has helicity

In the rest of this lecture I am going to use basics of QM and special relativity to argue that: [Any theory of massless spin 2 particle at low energies lead to GR.]

I follow lectures by Nima Arkani-Hamed, so refer to his lectures for much more transparent discussion..

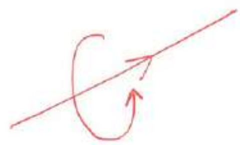
Concept of spin in QM

remember that in QM, we label our state with symmetry operators we have in our problem: In the particular case of spin, the label is associated to the SO(3) rotations: $j = 0, \frac{1}{2}, 1, \dots$

$m = -j, \dots, j \quad \# = 2j+1$

The assumption here is that we are in the rest frame of the particle.

For massless particles, there is no rest frame. The only thing that we can do is to rotate around the direction of propagation :



$\vec{J}\cdot\vec{k}$ is the operator

and this is an $SO(2)$ symmetry. The eigenvalues are half-integers : $0, \pm 1/2, \pm 1, \dots$

This label is known as helicity of the particle. If we take into account parity, then the degeneracy of this label will be : 2 $(e.g.\ +1 \leftrightarrow -1)$

j/h	$m \neq 0$	$m = 0$
0	1	1
1/2	2	2
1	3	2
3/2	4	2
2	5	2

there is a discontinuity in the number of d.o.f for $j/h \geq 1$ if we take massless limit !!

4 Fields and Redundancies

We use the concept of fields to make locality of interactions manifest. : which we need to make sure that theory is consistent with Lorentz invariance.

spin 0 $\longrightarrow$ $\phi(x)$ ✓

For spin 1 : $A_\mu(x)$ vector field

but this has 4 dof, while we only want 2!

It is possible to put constraints, which are Lorentz inv., to remove d.o.f:

for convenience, let's look at plane wave solutions

$$A_\mu = \varepsilon_\mu e^{ik\cdot x} \qquad \boxed{k_\mu \varepsilon^\mu = 0} \text{ removes } 1 \checkmark$$

$\partial_\mu A^\mu$

but there are no other possible constraints !!

The idea is to introduce the concept of redundancy.

States are not described by ε_μ, but they are described by equivalence class

$$\boxed{\{ \varepsilon_\mu \sim \varepsilon_\mu + \Lambda k_\mu \}}$$

i.e. polarization vectors that differ by a multiple of momentum describe the same state.

For spin 2 : $h_{\mu\nu}(x)$ we use tensor field

It has 10 d.o.f, we need 2!

Once again for a plane wave : $h_{\mu\nu} = \varepsilon_{\mu\nu} e^{ik\cdot x}$

$$\underset{4}{k_\mu \varepsilon^{\mu\nu} = 0} \quad , \quad \underset{1}{\eta_{\mu\nu} \varepsilon^{\mu\nu} = 0} \qquad 10 - 4 - 1 = 5$$

we take the equivalence class to be

$$\left\{ \varepsilon_{\mu\nu} \sim \varepsilon_{\mu\nu} + k_\mu \Lambda_\nu + k_\nu \Lambda_\mu \right\} \quad \text{with } k_\alpha \Lambda^\alpha = 0$$

$$5 - (4-1) = \boxed{2} \checkmark$$

- We can look at objects that are invariant under this transformations:

Spin 1 : $k_\mu \varepsilon_\nu - k_\nu \varepsilon_\mu \rightsquigarrow F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$

Spin 2 : you can show that the combination

is $\left(\partial_\mu \partial_\alpha h_{\nu\beta} - \cdots\cdots \right) \propto R^{(1)}_{\mu\nu,\alpha\beta}$ linearized Riemann !!

and then the e.o.m. (since we want 2nd derivative)

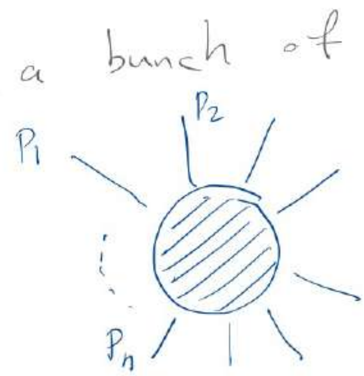
$$\partial_\mu F^{\mu\nu} = 0 \quad , \quad R^\alpha{}_{\mu,\alpha\nu} = 0 \quad \checkmark$$

- In the case of spin 2, it is possible to argue the existence of higher order self-int. and lead to full Einstein GR.

⑤ EP again

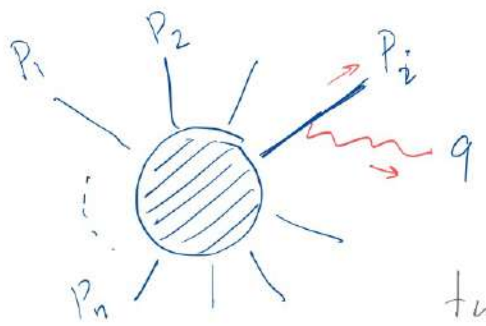
Here we follow an argument, due to Weinberg, about scattering involving a very low energy massless spin 1 or spin 2. A very low energy massless particle is called soft, so these are known as soft theorems.

Imagine a scattering process involving a bunch of particles, with amplitude



$$\mathcal{M}(p_1, \ldots, p_n)$$

Now, consider a similar process with an extra photo leg which is soft



$$\tilde{\mathcal{M}}(p_1, \ldots, p_n ; q)$$

We expect for soft photon these two amplitudes are related. In fact from the diagram we see that if it is attached to external legs we have an extra propagator

$$\frac{1}{(p_i+q)^2+m_i^2} = \frac{1}{2 p_i \cdot q}$$

which is large for small q. Notice that there are diagrams that the photon is attached to internal lines, but the internal lines are not on-shell so we don't get an enhancement.

We conclude that in the soft limit we have:

$$\lim_{q\to 0} \tilde{M} = M \sum_{i=1}^{n} \frac{e_i\, \varepsilon(q)\cdot p_i}{2p_i\cdot q} + \dots$$

photon polarization

there might be q here which we drop at leading order

Coupling of photon to i-th particle.

This is the standard coupling to scalars or Fermions. The magic is that, due to the redundancy: $\varepsilon_\mu \sim \varepsilon_\mu + q_\mu$ and therefore, if we replace in any amplitude $\varepsilon(q)\to q$ we must get zero.

$$\boxed{\sum_i e_i = 0}$$

We discover electric charge conservation !!

Let's repeat the same logic for graviton:

$$\lim_{q\to 0} \tilde{M} = M \sum_{i=1}^{n} \frac{k_i\, \varepsilon_{\mu\nu} p_i^\mu p_i^\nu}{2p_i\cdot q}$$

graviton polarization.

there is q dependence which is dropped.

Coupling of i-th particle to graviton.

Once again the amplitude must not change if we replace

$\varepsilon_{\mu\nu} \longrightarrow \varepsilon_{\mu\nu} + q_\mu \Lambda_\nu + q_\nu \Lambda_\mu$. Therefore we have

$$0 = \sum_i k_i \frac{\cancel{2(q \cdot P_i)}(\Lambda \cdot P_i)}{\cancel{2 P_i \cdot q}}$$

Or in other words, since Λ is arbitrary, $\boxed{\sum_i k_i P_i^\mu = 0}$

There is already one linear constraint on the external momenta due to momentum conservation: $\sum_i P_i^\mu = 0$. Putting another one implies that the theory is forbidding some kinematics!

The only way out is that $\boxed{k_i = k}$: the coupling to graviton is <u>universal</u>.

This is the manifestation of EP, in scattering process: as a consequence of consistency of QM + SR for massless spin 2 particle.