Gravitational Waves

One of the most remarkable prediction of Maxwell equations is EM waves. Similarly GW is among the most important predictions of GR.

- The main difference is that Einstein equations are non-linear (since gravity has energy and backreacts on itself) $\Rightarrow$ It is much more difficult to find exact wave solutions.
- In most cases, We are at extremely large distances from the source, or the source is very weak. Then we can linearize the equations. then practically everything is the same as EM with an extra index.

1) Linearized Einstein equations

Close to Minkowski metric : $g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{h_{\mu\nu}(x)}_{\text{small}}$

The inverse metric is $\quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \frac{1}{2} h^{\mu\alpha} h_\alpha{}^\nu + \underbrace{\dots}_{\text{higher order}}$

$h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$

- Notice that we are thinking of $h_{\mu\nu}(x)$ as a tensor field living in Minkowski background: Raise/lower indices with η, etc.
- Christoffel symbols: $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\lambda}\left[\partial_\alpha h_{\lambda\beta} + \partial_\beta h_{\lambda\alpha} - \partial_\lambda h_{\alpha\beta}\right] + \cdots$
- Riemann and Ricci: $R \sim \partial\Gamma - \partial\Gamma$ $\quad (\Gamma\Gamma \text{ is } \mathcal{O}(h^2))$

  so we only have 2nd derivatives of h:

$$\boxed{R_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\sigma h^\sigma{}_\nu + \partial_\nu\partial_\sigma h^\sigma{}_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right) + \cdots}$$

$h \equiv \eta^{\alpha\beta}h_{\alpha\beta}$ $\qquad$ $\eta^{\alpha\beta}\partial_\alpha\partial_\beta$

and

$$\boxed{R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h + \cdots}$$

and therefore: $G_{\mu\nu} \equiv R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} + \cdots$

- A usefull object to define: $\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h}$

  this is known as trace-reversed field

  since $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -h$ ✓

  We can do this operation twice to get back the original object: $\bar{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} = h_{\mu\nu}$.

  Notice that going from $R_{\mu\nu}$ to $G_{\mu\nu}$ is also trace-reverse operation.

- In terms of $\bar{h}$ we have, Einstein equations are

$$\boxed{\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \partial_\mu \partial_\sigma \bar{h}^\sigma{}_\nu - \partial_\nu \partial_\sigma \bar{h}^\sigma{}_\mu = -16\pi G\, T_{\mu\nu}}$$

- Doing the trace-reverse operation, or equivalently start from trace-reversed Einstein equations ($R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$) we obtain

$$\boxed{\Box h_{\mu\nu} - \partial_\mu \partial_\nu h + \partial_\mu \partial_\sigma h^\sigma{}_\nu + \partial_\nu \partial_\sigma h^\sigma{}_\mu = -16\pi G\, \bar{T}_{\mu\nu}}$$

$\bar{T}_{\mu\nu} \to T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T$, $T \to \eta^{\alpha\beta} T_{\alpha\beta}$

- A few important comments:
  - Remember that full equations satisfy $\nabla_\mu G^{\mu\nu} = 0$
    at linear order in h, this is equal to $\partial_\mu G^{\mu\nu(1)} = 0$
    which can be check explicitly above.
    For consistency at this order we must have $\boxed{\partial_\mu T^{\mu\nu} = 0}$
    So in the expression for $T_{\mu\nu}$ we go at zeroth order.
    Otherwise, we cannot have conservation: $T^{(0)}_{\mu\nu}$
  - As mentioned before, $h_{\mu\nu}(x)$ is a tensor field in Minkowski space: It transform under Lorentz transformation

$$h'_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta h_{\alpha\beta}$$

Consistent with the fact that $T^{(0)}_{\mu\nu}$ on the RHS has the same property.

- There is a gauge redundancy in the equation above

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \qquad \forall\ \xi_\mu(x)$$

leaves the equation invariant (check explicitly), Notice RHS is invariant trivially as there is no h.

This is the linearized general coordinate transformation Co-variance of the full equations:

$$g^{\mu\nu\prime} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta}\, g^{\alpha\beta}$$

choose: $x'^\mu = \underbrace{x^\mu - \xi^\mu(x)}_{\text{small} \sim O(h)} \longrightarrow \dfrac{\partial x'^\mu}{\partial x^\alpha} = \delta^\mu_\alpha - \partial_\alpha \xi^\mu$

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x) + \cdots$$

$$g^{\mu\nu\prime}(x') = \eta^{\mu\nu} - h'^{\mu\nu}(x) + \cdots$$

no need to distinguish between x and x'

we obtain: $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

Notice that, while $G_{\mu\nu}$ is covariant under general coord. trans. $G^{(1)}_{\mu\nu}$ is invariant, under its linearized form, since itself is $O(h)$.

- One can show that the above combination is unique if we require gauge invariance. (HW)

- A convenient gauge choice is the linearized harmonic gauge

$$0 = g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta} = \boxed{\partial_\alpha h^{\alpha\mu} - \frac{1}{2}\partial^\mu h + \cdots = \partial_\alpha \bar{h}^{\alpha\mu} + \cdots}$$

this is similar to Lorentz gauge $(\partial_\mu A^\mu = 0)$ in EM.

We should make sure that this choice is in fact possible:

$$0 = \partial_\alpha \bar{h}^{\alpha\mu\,\prime} = \partial_\alpha \bar{h}^{\alpha\mu} + \Box \xi^\mu \Rightarrow \underline{\Box \xi^\mu = -\partial_\alpha \bar{h}^{\alpha\mu}}$$

We can solve this, in principle, using Green function method:

$$\xi^\mu(x) = \int d^4x' \, G(x-x')\left(-\partial_\alpha \bar{h}^{\alpha\mu}\right)(x') \qquad \text{with} \qquad \Box G(x) = \delta^4(x)$$

By solving this equation (there could be many solutions, pick one)

we can demand the Lorentz gauge.

In this gauge Einstein equations simplify significantly:

$$\boxed{\begin{aligned} \Box \bar{h}_{\mu\nu} &= -16\pi G\, T_{\mu\nu} \\ \text{or} \quad \Box h_{\mu\nu} &= -16\pi G\, \bar{T}_{\mu\nu} \end{aligned}}$$

notice the conservation $\partial_\mu T^{\mu\nu} = 0$, is consistent in this gauge

- Notice that after gauge fixing there are $10 - 4 = 6$ independent d.o.f, with 6 independent equations.

(2) Quadratic Action

Instead of linearizing equations, we could look at the quadratic action. Notice that the linear term in the action is proportional to the background, i.e. Minkowski, equation of motion, and is therefore zero.

From the EH term

$$S_{EH} = \frac{1}{16\pi G}\int d^4x \underbrace{\sqrt{-g}}_{(HW)\ (1+\frac{1}{2}h+\cdots)}\ \underbrace{g^{\mu\nu}}_{(\eta^{\mu\nu}-h^{\mu\nu}+\cdots)}\ \underbrace{R_{\mu\nu}}_{(R^{(1)}_{\mu\nu}+R^{(2)}_{\mu\nu}+\cdots)}$$

$R^{(2)}_{\mu\nu}$: (HW)

$$= \frac{-1}{64\pi G}\int d^4x \left[(\partial_\mu h_{\alpha\beta})^2 - (\partial_\mu h)^2 - 2\left(\partial_\alpha h^{\alpha\mu}\right)^2 + 2\,\partial_\mu h\, \partial_\alpha h^{\alpha\mu}\right]$$

- Relative signs could have been easily fixed using gauge inv. (HW)
- Imposing a gauge at the level of the action (for QFT) is to add a term $\frac{2}{\xi}\left(\partial_\alpha h^{\alpha\mu} - \frac{1}{2}\partial^\mu h\right)^2$ to the action. This is the analog of R_ξ gauge in EM when we add $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ to the action.

Choosing ($\xi = 1$) simplifies a lot: $S_{EH} = \frac{-1}{64\pi G}\int d^4x \left[(\partial_\mu h_{\alpha\beta})^2 - \frac{1}{2}(\partial_\mu h)^2\right]$

Feynman gauge

- For the matter part we simply get the EMT from the definition:

$$S_M = \frac{1}{2}\int d^4x \, T_{\mu\nu} h^{\mu\nu} + \dots$$

it's enough to stop at linear order here

Then it is easy to obtain the linearized equation derived before by varying with respect to $h_{\mu\nu}$.

3 Plane Wave Solutions

In a source free region we can look for plane wave solutions of the wave equation

$$\Box h_{\mu\nu} = 0 \qquad \text{with} \qquad \partial_\alpha \bar{h}^{\alpha\mu} = 0$$

The ansatz is $h_{\mu\nu}(x) = \varepsilon_{\mu\nu}(k)\, e^{ik\cdot x}$ $\quad \eta_{\alpha\beta} k^\alpha x^\beta$

polarization tensor

This is a solution if: $\boxed{k^2 = 0}$ and $\boxed{k_\mu \varepsilon^{\mu\nu} - \frac{1}{2} k^\nu \varepsilon = 0}$

GW propagates @ speed of light.

4 constraints on 10 components of $\varepsilon_{\mu\nu}$.

• There is a residual gauge redundancy : Harmonic gauge condition doesn't completely fix the gauge :

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \qquad \text{with } \Box \xi_\mu = 0$$

for a generic ξ_μ, this will take us away from harmonic gauge, but with $\Box \xi_\mu = 0$, $\partial_\alpha \bar{h}^{\alpha\mu} = 0$ remains true.

[remember that when choosing gauge we had to solve $\Box \xi_\mu = -\partial^\alpha \bar{h}_{\alpha\mu}$, this doesn't have unique solution ; we can add to it any solution of $\Box \xi_\mu = 0$]

• This redundancy is always there. But in the source-free case we can use this to get rid off some components of $h_{\mu\nu}$:

(1) Turn on ξ_0 to remove the trace h

(2) Turn on ξ_i to remove h_{0i}

We have 4 function redundancies we knock out 4 functions.

We will see explicitly how would that be possible.

$$\Rightarrow \quad 10 - 4 - 4 = 2 \text{ propagating d.o.f}$$

Then the harmonic gauge condition is : $\partial_\alpha h^{\alpha\mu} = 0$

$\mu = 0 \qquad \partial_t h^{00} = 0 \quad \xrightarrow{\text{plane wave}} \quad \mathcal{E}_{00} = 0 \qquad \mathcal{E}_{0i} = 0$ $h_{0i}=0$

$\mu = i \qquad \partial_j h^{ji} = 0 \quad \longrightarrow \quad \underline{k_i \mathcal{E}^{ij} = 0} \qquad \underline{\mathcal{E}^i{}_i = 0}$ $h=0$

transverse traceless

This is also known as traceless-transverse (TT) gauge.

- If we choose the wave vector to be $k^\mu = (k, 0, 0, k)$ propagation along z-axis :

$$\mathcal{E}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathcal{E}_+ & \mathcal{E}_\times & 0 \\ 0 & \mathcal{E}_\times & -\mathcal{E}_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

or for a super-position of waves

$$h_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

function of $(t-z)$

diagonal part is known as plus-polarization.
off- " " " " " cross-polarization.

- Notice that going from harmonic to TT gauge is similar to EM when we start from Lorentz gauge and then go to radiation (Coulomb) gauge :

$$\partial_\mu F^{\mu\nu} = -J^\nu \quad \xrightarrow{\partial_\alpha A^\alpha = 0} \quad \Box A^\nu = -J^\nu$$

residual gauge trans: $A_\mu \to A_\mu + \partial_\mu \chi$ with $\Box \chi = 0$ (to remain $\partial_\alpha A^\alpha = 0$)

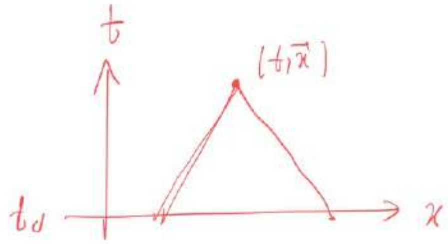
The hope is that using this extra redundancy we can have $A_0' = 0$. Notice that we should not expect this to happen arround points with $J_0 \neq 0$: The reason is that this is not consistent with $\Box A_0' = -J_0$.

More explicitly, $A'_\mu = A_\mu + \partial_\mu \chi$. Let's choose χ such that $0 = A_0 + \partial_t \chi$ which using $\Box \chi = 0$ and $\partial_\alpha A^\alpha = 0$ implies $\nabla^2 \chi = -\vec{\nabla} \cdot \vec{A}$. If we impose

1) $\nabla^2 \chi = -\vec{\nabla} \cdot \vec{A}$ and 2) $\partial_t \chi = -A_0$ as the initial data $(t=t_0)$, we can solve $\Box \chi = 0$ and find χ everywhere.

Then $A'_\mu = A_\mu + \partial_\mu \chi$ satisfies $\Box A'_\mu = -J_\mu$, with $A'_0 = 0$ at $t=t_0$

$$A'_0(x) = -\int_{t_0}^{\infty} dt' \int d^3\vec{x}' \; G_{ret}(t-t', \vec{x}-\vec{x}') \, J_0(x')$$



so if there is a source-free region

we can take $A_0 = 0$.

Similarly for linearized gravity:

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \qquad \text{with} \quad \Box \xi_\mu = 0 \quad \text{to maintain } \partial_\alpha \bar{h}^{\alpha\mu} = 0$$

We solve these four wave equations with the following initial data:

$$\begin{cases} -\partial_t \xi_0 + \partial_i \xi_i + \frac{1}{2} h = 0 \\ \partial_t \xi_i + \partial_i \xi_0 + h_{0i} = 0 \\ -\nabla^2 \xi_0 + \partial_t \partial_i \xi_i + \frac{1}{2} \partial_t h = 0 \\ \nabla^2 \xi_i + \partial_t \partial_i \xi_0 + \partial_t h_{0i} = 0 \end{cases}$$

fixes ξ_μ and $\partial_t \xi_\mu$ initial time

Then in a source free region: $\boxed{h' = h'_{0i} = 0}$

- If this is all very confusing, the bottom line is that the only propagating degrees of freedom, is the transverse and trace-less part of the spatial metric.

To see this explicitly write the linearized Einstein equations with a source:

$$h_{00}, \quad h_{0i} = \partial_i w + w_i^T \rightarrow \partial_i w_i^T = 0$$

$$h_{ij} = \Psi \delta_{ij} + \left(\partial_i \partial_j - \tfrac{1}{3}\delta_{ij}\right)\chi + \left(\partial_i C_j^T + \partial_j C_i^T\right) + h_{ij}^{TT}$$

$\partial_i C_i^T = 0$ $\qquad$ $\partial_i h_{ij}^{TT} = 0$, $\quad h^{TT\,i}{}_i = 0$

these are like Helmholtz decomposition of EM and they are unique

Then we see that the only part which satisfies a wave equation is h_{ij}^{TT} : $\Box h_{ij}^{TT} \propto T_{ij}^{TT}$

Notice that this way is the most natural way in Cosmology.

- From the above plane wave solution moving in z-direction we can generate other directions by a rotation:

$$h'_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta h_{\alpha\beta}$$

$$\rightarrow h'_{ij} = R_i{}^m R_j{}^n h_{mn}$$

what if we rotate around $\hat{z}$: the direction doesn't change but polarization tensors rotate into each other:

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

2d transverse plane.

$$\varepsilon'_+ = R^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R = \cos 2\theta\, \varepsilon_+ + \sin 2\theta\, \varepsilon_\times$$

$$\varepsilon'_\times = R^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} R = -\sin 2\theta\, \varepsilon_+ + \cos 2\theta\, \varepsilon_\times$$

$$e_\pm \equiv \varepsilon_+ \mp i\, \varepsilon_\times = \begin{bmatrix} 1 & \mp i \\ \mp i & -1 \end{bmatrix} \Longrightarrow \boxed{e'_\pm = e^{\pm 2i\theta} e_\pm}$$

These are the analog of circularly polarized waves in EM. A field that transforms like $\psi' = e^{ih\theta}\psi$ after rotation around the propagation direction is called helicity h $\Rightarrow$ GW is composed of helicity ± 2 waves.

- It is useful to write $e^{\pm}_{\mu\nu}$ in terms of $e^{\pm}_{\mu} = (0, 1, \mp i, 0)$ of EM (which has helicity ± 1). It is easy to see that:

$$\boxed{e^{\pm}_{\mu\nu} = e^{\pm}_{\mu} e^{\pm}_{\nu}}$$

4) Interaction with test particles

The metric in the presence of GW takes the following form:

function of $(t-z)$

$$ds^2 = (\eta_{\mu\nu} + h^{TT}_{\mu\nu})\, dx^\mu dx^\nu$$

$$= -dt^2 + (1+h_+)dx^2 + (1-h_+)dy^2 + 2h_\times dx\,dy + dz^2$$

We can try to see how a free particle is affected:

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0$$

To see the effect of GW we must look at <u>proper distances</u> (rather than coordinate distances)

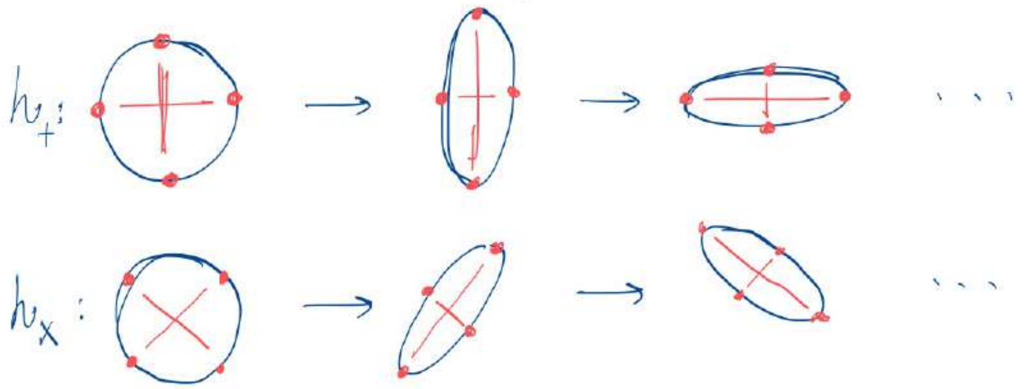
For a coordinate separation $\xi^\mu = (0, \vec{\xi})$

the proper distance is :

$$S = \sqrt{ds^2} = \sqrt{\vec{\xi}^{\,2} + h_{ij}^{TT}\xi^i\xi^j} = |\vec{\xi}|\left(1 + \frac{1}{2} h_{ij}^{TT} \frac{\xi^i\xi^j}{\xi^2} + \dots\right)$$

$$\frac{\Delta S}{S} = \frac{1}{2} h_{ij}^{TT} \frac{\xi^i\xi^j}{\vec{\xi}^{\,2}} = \boxed{\frac{1}{2} h_+ \frac{\xi_x^2 - \xi_y^2}{\xi^2} + h_\times \frac{\xi_x\xi_y}{\xi^2}}$$

Here I am assuming $\xi \ll \lambda$ wavelength of GW.



- Notice that while Ricci is zero Riemann is not and it is equal to Weyl.
- We can describe this behavior in "proper detector" frame. Which means that by doing a coordinate transformation (gauge transformation), we make the metric locally flat

like seen from a freely falling observer.

We have seen that it is only possible to do this locally but at some point deviations must show up:

$$ds^2 = -dt^2\left(1 + R_{0i0j}\, x^i x^j\right) - 2\, dt\, dx^i \left(\frac{2}{3} R_{0jik}\, x^j x^k\right)$$
$$+ dx^i dx^j \left(\delta_{ij} - \frac{1}{3} R_{ikjl}\, x^k x^l\right)$$

in the vicinity of $\vec{x} = 0$. This is known as Fermi Normal Coord.

Up to linear terms in $\vec{x}$, it is flat; deviation start from x^2 which are important if $x^2/L^2 \sim 1$: L characteristic length scale of background (curvature, in this case λ of GW).

- Notice that to change to this coordinate: $h_{\mu\nu}^{TT} \longrightarrow h'_{\mu\nu}$ but Riemann is invariant at linear order so we can use the previous form $h_{\mu\nu}^{TT}$ to calculate Riemann.

Geodesic deviation:

$$\frac{d^2 \xi^i}{d\tau^2} = -\underbrace{R^i{}_{0j0}}_{-\frac{1}{2}\ddot{h}_{ij}^{TT}}\, \xi^j$$

for particles at rest.

which implies : $\boxed{\ddot{\xi}^i = \frac{1}{2}\ddot{h}_{ij}^{TT}\xi^j}$

like a tidal force !!

This results in the same conclusion as before, written in a different coord.

5) Energy, Momentum and angular momentum of GW

We can extract this by looking at the energy-momentum tensor of GWs, using the quadratic action we wrote earlier:

From Noether's theorem we have

$$t^{\mu\nu} \equiv - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \partial^\nu \varphi_a + \eta^{\mu\nu}\mathcal{L}$$

$\varphi_a \to h_{\alpha\beta}$

After a long