which implies : $\boxed{\ddot{\xi}^i = \frac{1}{2}\ddot{h}_{ij}^{TT}\xi^j}$ → like a tidal force !!

This results in the same conclusion as before, written in a different coord.

5) Energy, Momentum of GW

We can extract this by looking at the energy-momentum tensor of GWs, using the quadratic action we wrote earlier:

From Noether's theorem we have

$$t^{\mu\nu} \stackrel{!}{=} -\frac{\partial \mathcal{L}}{\partial(\partial_\mu h_{\alpha\beta})}\partial^\nu h_{\alpha\beta} + \eta^{\mu\nu}\mathcal{L}$$

which conservation is ensured: $\partial_\mu t^{\mu\nu} = 0$

This has two problems: 1) It is not symmetric in $\mu\nu$ the canonical Noether current. This is true for any Noether current that we can add to it a term $\partial_\sigma \theta^{\sigma\mu\nu}$ with θ being anti-symmetric in $\sigma \leftrightarrow \mu$. Then this is conserved by itself $\partial_\mu\partial_\sigma\theta^{\sigma\mu\nu} = 0$, and since this is a total derivative doesn't affect conserved total energy and momentum.

One can show that in a Lorentz inv. theory we can always Symmetrize the energy-momentum tensor by adding such terms.

Equivalently, we could put our action on some background metric $\bar{g}_{\mu\nu}$ and $\partial_\mu \to \bar{\nabla}_\mu$, then calculate $t_{\mu\nu} \sim \left.\frac{\delta S_{EH}}{\delta \bar{g}^{\mu\nu}}\right|_{\bar{g}=\eta}$ to get a symmetric result.

An equivalent way of doing this is to start from full Einstein equations, and decompose the metric as

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ but not assume that h is small for now, but goes to zero at infinity

Then re-write the equation as follows:

$$G^{(1)}_{\mu\nu} = 8\pi G\left[T_{\mu\nu} + t_{\mu\nu}\right] \quad \text{with} \quad t_{\mu\nu} = \frac{-1}{8\pi G}\left[G_{\mu\nu} - G^{(1)}_{\mu\nu}\right]$$

linear in h $\sim -\frac{1}{2}\Box\bar{h}_{\mu\nu} + \dots$

pseudo-tensor

Then $t_{\mu\nu}$ will be the energy-momentum of gravity

For weak GW we can truncate @ $O(h^2)$

$$\boxed{t_{\mu\nu} \simeq \frac{-1}{8\pi G} G^{(2)}_{\mu\nu} + \dots}$$

higher order terms means that there is self-interaction.

and $T_{\mu\nu} + t_{\mu\nu}$ will be the total energy-momentum tensor.

- Notice that by linearized Bianchi $\partial_\mu G^{(1)\mu\nu} = 0$

and therefore $\partial_\mu (T^{\mu\nu} + t^{\mu\nu}) = 0$

So we can define total energy-momentum of spacetime

$$P^\mu \equiv \int d^3x \, \underbrace{(T+t)^{0\mu}}_{\text{density of energy-momentum}}$$

$(T+t)^{i\mu}$ is the flux along $\hat{x}^i$

Similar to QFT we can define total angular momentum:

$$\mathcal{M}^{\mu\nu\lambda} \equiv (T+t)^{\mu\nu} x^\lambda - (T+t)^{\mu\lambda} x^\nu$$

then it is conserved: $\partial_\mu \mathcal{M}^{\mu\nu\lambda} = 0$. The conserved charge

$$J^{ij} = \int d^3x \, \mathcal{M}^{0ij}$$

is total angular momentum in (ij) plane and $\mathcal{M}^{kij}$ is its flux in $\hat{x}^k$ direction.

2) This is not gauge invariant: $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

Technically this is easy to see, because we know that by going to the free-fall coord. system we can make h and ∂h vanish. And $t_{\mu\nu} \sim (\partial h)^2$ so we can make it zero locally.

This is in contranst with EM that its $T_{\mu\nu}$ after symmetrization is gauge invariant.

- It should be noted that if we are interested in the total energy, momentum, etc. the result will be invariant under coordinate transformations that vanish at infinity. So there are well-defined notions at least for asymptotically flat spacetime.

- For the case of GW, we can use averaging to define a well-defined notion of energy-momentum which is almost local.

- Notice that taking an average to calculate the energy is what typically we do for waves:

Example : $$\phi(t,z) = A\cos(\omega(t-z))$$

then the energy of the wave is

$$\frac{1}{2}\left(\dot{\phi}^2 + (\partial_z\phi)^2\right) = A^2\omega^2 \sin^2(\omega(t-z))$$

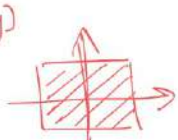
but what we care about is the average over a cycle

$$E = \left\langle \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\partial_z\phi)^2 \right\rangle = \frac{1}{T}\int_0^T dt\, A^2\omega^2 \sin^2(\omega(t-z)) = \frac{1}{2}A^2\omega^2$$

or similarly with an spatial average.

- Therefore we define the EMT of GW as

$$\boxed{t_{\mu\nu} = \frac{-1}{8\pi G}\left\langle G^{(2)}_{\mu\nu} \right\rangle} \rightarrow \int d^4y\, \underbrace{W(x-y)}_{\text{window function}} G_{\mu\nu}(y)$$



where we assume the average is over a few cycles

Notice that this is a coarse-grained definition.

what convolving with a Window function does, is to focus on the "low-frequency" components of the expressions : For GW this is what contributes to the background.

Averaging has the advantage that we can do integration by parts (up to small corrections).

After a long calculation in TT gauge we obtain

$$\boxed{t_{\mu\nu} = \frac{1}{32\pi G} \left\langle \partial_\mu h^{TT}_{\alpha\beta} \, \partial_\nu h^{TT\alpha\beta} \right\rangle}$$

this is far out side of matter such that we can go to TT.

- Notice that it is conserved on the equation of motion:

$$\partial_\mu t^{\mu\nu} = \frac{1}{32\pi G} \left\langle \underbrace{\Box h}_{0} \, \partial^\nu h + \underbrace{\partial^\mu h \, \partial_\mu \partial^\nu h}_{=0 \text{ after int. by part}} \right\rangle = 0$$

And it is gauge invariant: $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

$$t_{\mu\nu} \to t_{\mu\nu} + \frac{2}{32\pi G} \left\langle \partial_\mu h_{\alpha\beta} \left(\partial_\nu \partial^\alpha \xi^\beta + \partial_\nu \partial^\beta \xi^\alpha \right) \right\rangle$$

after by part and use harmonic condition $\partial_\alpha h^{\alpha\beta} = 0$

$$= t_{\mu\nu}$$

- For the plane wave solution

$$h_{\mu\nu} = \varepsilon_{\mu\nu} e^{ik\cdot x} + cc \rightarrow \partial_\mu h_{\alpha\beta} = ik_\mu \varepsilon_{\alpha\beta} e^{ik\cdot x} + cc$$

$$\Longrightarrow \quad \boxed{t_{\mu\nu} = \frac{k_\mu k_\nu}{16\pi G} \, \varepsilon_{\alpha\beta} \, \varepsilon^{\alpha\beta *}}$$

terms like $e^{2ik\cdot x}$ will drop out.

for a wave moving along z direction: $\varepsilon_{\alpha\beta} = \begin{bmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{bmatrix}$

then: $$t_{\mu\nu} = \frac{k_\mu k_\nu}{8\pi G} \left(|h_+|^2 + |h_\times|^2 \right)$$ with $k_\mu = (-k, 0, 0, k)$

Notice that we can now rotate the coordinates and the form is true for generic k_μ.

(6) Emission of GW

So far we have only discussed the homogeneous wave equation. Now we consider a source term:

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu} \qquad \text{with} \quad \partial_\alpha \bar{h}^{\alpha\mu} = 0$$

Using the Green function of the scalar wave equation

$$\Box_x G_{\text{ret}}(x-x') = \delta^4(x-x') \qquad \text{with} \quad t<t' \to G_{\text{ret}}(x-x') = 0$$

We have:

$$G_{\text{ret}}(x-x') = \underbrace{\frac{-1}{4\pi|\vec{x}-\vec{x}'|}}_{\text{usual damping } 1/r}\ \underbrace{\delta\left((t-t') - |\vec{x}-\vec{x}'|\right)}_{\text{propagation with speed of light.}}\ \underbrace{\theta(t-t')}_{\text{retardation}}$$

Then the solution is

$$t_{\text{ret}} \equiv t - |\vec{x}-\vec{x}'|$$

$$\boxed{\bar{h}_{\mu\nu}(x) = 4G\int d^3x'\ \frac{T_{\mu\nu}(t-|\vec{x}-\vec{x}'|, \vec{x}')}{|x-\vec{x}'|}}$$

Coulomb law with retarded time.

- Derivation of Green function:

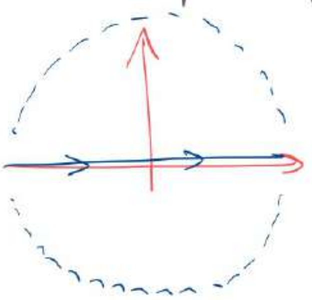
$$(-\partial_t^2 + \vec{\nabla}^2)\, G(x) = \delta^4(x)$$

$$\hookrightarrow \quad -p^2 G(p) = 1$$

$$\begin{cases} G(x) = \displaystyle\int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x}\, G(p) & p\cdot x = p_\mu x^\mu \\ G(p) = \displaystyle\int d^4x\ e^{-ip\cdot x}\, G(x) \end{cases}$$

$$G(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \frac{1}{p^{02} - \vec{p}^{\,2}} = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \frac{e^{-ip^0 t}}{p^{02} - \vec{p}^{\,2}}$$

for The temporal part we can use Cauchy theorem :



there is an ambiguity for the location of the poles $\frac{1}{p^{02} - \vec{p}^{\,2}}$; $p^0 = \pm |\vec{p}|$ since they are on the integration contour: we remove this ambiguity by adding a small imaginary part $\pm i\epsilon$ (similar to Feynman $i\epsilon$)

How to decide the sign? We demand that for $t<0$ the integral vanishes: retarded Green function

It is easy to see that the correct prescription is $\frac{1}{(p^0+i\epsilon)^2 - \vec{p}^{\,2}}$ so both poles are in the lower half plane.

For $t>0$ we must close the contour in LHP

$$\int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \frac{e^{-ip^0 t}}{(p^0+i\epsilon)^2 - \vec{p}^{\,2}} = \frac{\overset{\text{clockwise}}{-2\pi i}}{2\pi} \left[\frac{e^{-i|\vec{p}|t}}{2|\vec{p}|} + \frac{e^{i|\vec{p}|t}}{-2|\vec{p}|} \right] \qquad t>0$$

$$\rightarrow \quad G(x) = \frac{1}{2i} \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{x}}}{|\vec{p}|} \left(e^{-i|\vec{p}|t} - e^{i|\vec{p}|t} \right) \theta(t)$$

$$\overset{\text{integrals over angle}}{=} \frac{1}{2i} \int_0^{\infty} \frac{d|\vec{p}|}{(2\pi)^2} \frac{1}{i|\vec{x}|} \left(e^{i|\vec{p}||\vec{x}|} - e^{-i|\vec{p}||\vec{x}|} \right) \left(e^{-i|\vec{p}|t} - e^{i|\vec{p}|t} \right) \theta(t)$$

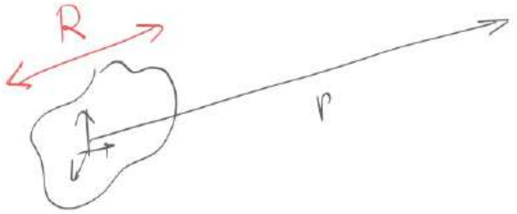
$$= \frac{-\theta(t)}{4\pi|\vec{x}|}\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d|\vec{p}|}{2\pi}\left(e^{i|\vec{p}||\vec{x}|}-e^{-i|\vec{p}||\vec{x}|}\right)\left(e^{-i|\vec{p}|t}-e^{i|\vec{p}|t}\right)$$

$$= \frac{-\theta(t)}{4\pi|\vec{x}|}\left[\delta(t-|\vec{x}|) - \cancelto{0}{\delta(t+|\vec{x}|)}\right]$$

$$= \boxed{\frac{-\theta(t)\,\delta(t-|\vec{x}|)}{4\pi|\vec{x}|}}$$

- The expression for $h_{\mu\nu}$, is conceptually nice, but is difficult to calculate in most generality.
- Typically sources are <u>localized</u> and we are looking at GW very far from them:

$$|\vec{x}-\vec{x}'| \simeq \underset{\downarrow\, |\vec{x}|}{r} - \vec{x}'\cdot\hat{n} + \ldots \qquad \hat{n} = \frac{\vec{x}}{r} \ \ \uparrow \text{direction of looking}$$



$$\boxed{r \gg R}$$

notice that is not an approximation on the source: just a choice of the location.

- There is another scale in the problem which is the time scale of the variation of the source.

This is relevant since we have two appearance of $|\vec{x}-\vec{x}'|$.

- It is much easier to see this if we go to time Fourier space for the source

$$T_{\mu\nu}(t,\vec{x}) = \int \frac{d\omega}{2\pi} e^{-i\omega t} T_{\mu\nu}(\omega,\vec{x})$$

Then the solution can be written as:

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4G\int \frac{d\omega}{2\pi} d^3x' \frac{e^{-i\omega(t-|\vec{x}-\vec{x}'|)}}{|\vec{x}-\vec{x}'|} T_{\mu\nu}(\omega,\vec{x}')$$

Then we can imagine two regimes:

1- <u>Near Zone</u> $\quad R \ll r \ll \omega^{-1} \rightarrow$ typical ωs of the source.

then inside the integrand corrections of $\frac{1}{|x-x'|}$ will be $\left(\frac{x'}{r}\right)$

but from the exponential $\sim \omega x' \sim \omega R \ll \frac{R}{r} \sim \frac{x'}{r}$

So we keep the exponent at leading order and expand $\frac{1}{|x-x'|}$

$$\bar{h}_{\mu\nu}(x) = 4G\int \frac{d\omega}{2\pi} e^{-i\omega(t-r)} \sum_L \frac{\hat{n}^L}{r^{L+1}} \int d^3x' \, T_{\mu\nu}(\omega,x') \, x'_L$$

$$= \boxed{4G\sum_{\ell=0}^{\infty} \frac{\hat{n}_L}{r^{\ell+1}} \int d^3x' \, T_{\mu\nu}(t-r,x') \, x'^L}$$

multipole expansion of electrostatic at retarded time.

multi-index notation: $\quad \hat{n}^L = \hat{n}^{i_1}\hat{n}^{i_2}\dots\hat{n}^{i_\ell} \qquad x'^L = x'^{i_1}\dots x'^{i_\ell}$

2 - Far Zone (Wave Zone) $\quad R \ll \omega^{-1} \ll r$

We keep the denominator at leading order but go to higher orders in the numerator.

$$\bar{h}_{\mu\nu}(x) = 4G \int \frac{d\omega}{2\pi} \, \frac{e^{-i\omega(t-r)}}{r} \sum_{\ell=0}^{\infty} \frac{(-i\omega)^\ell \, \hat{n}_L}{\ell!} \int d^3x' \, T_{\mu\nu}(\omega, \vec{x}') \, x'^L$$

$$= \boxed{\frac{4G}{r} \sum_{\ell=0}^{\infty} \frac{\hat{n}_L}{\ell!} \int d^3x' \, \partial_t^\ell \, T_{\mu\nu}(t-r, \vec{x}') \, x'^L}$$

- Notice that in this expression we only expanded in powers of (ωR) and kept $\frac{R}{r} \ll \omega R$ and therefore in the exponent we set: $\exp\left[-i\omega(t-|\vec{x}-\vec{x}'|)\right] \simeq e^{-i\omega(t-r)} \, e^{-i\omega x' \cdot \hat{n}}$

- Notice that all of these terms drop as $1/r$ which is the character of the waves that carry energy from the source to infinity.

- An important assumption which was implicit: $\boxed{R \ll \omega_s^{-1}}$

We can interpret this more properly as

$$R \ll c\underbrace{\tau_s}_{\text{time scale}} \rightarrow \boxed{\frac{v}{c} \ll 1}$$

This must be true for sources that move slowly $\sim$ Non-relativistic systems

If the system is not slow, then we can still go very far such that $R, \omega^{-1} \ll r$ and then expand the denominator to get a wave zone.

$$\bar{h}_{\mu\nu}(t,\vec{x}) = \frac{4G}{r}\int d^3x' \left[T_{\mu\nu}(t-r+\vec{x}'\cdot\hat{n}, \vec{x}') + \underbrace{\frac{\hat{x}'\cdot\hat{n}}{r}}_{\text{suppressed for large } r} T_{\mu\nu}(t-r+\hat{x}'\cdot\hat{n}, \vec{x}') + \cdots \right]$$

Notice that we have not expanded the argument of $T_{\mu\nu}$ since time variation of the source could be large.

- Since we are outside of the source (slow or not) we use the extra gauge to go to TT gauge. We can do this by a projection tensor:

  - $\Pi_{ij} \equiv \delta_{ij} - n_i n_j$ projection orthogonal to n_i
    $\Pi \cdot \Pi = \Pi$ $n_i \Pi_{ij} = 0$

  - $\Lambda_{ij,kl} \equiv \Pi_{ik}\Pi_{jl} - \frac{1}{2}\Pi_{ij}\Pi_{kl}$ projector, transverse to n_i
    $\Lambda \cdot \Lambda = \Lambda$ traceless in (ij) and (kl)

Therefore if we have a generic spatial tensor

$h^{TT}_{ij} = \Lambda_{ij,kl} h_{kl}$, the resulting tensor will be trace-less and transverse with respect to n_i.

Since the expressions above are only a function of r:

$$\partial_i \bar{h} \propto \frac{x^i}{r} \partial_r \bar{h} \propto \hat{n}_i$$

Therefore in our projection tensor, $\hat{n}_i$ will be the direction of looking!

So for generic sources we have TT gauge expression:

$$\boxed{h_{ij}^{TT}(t,\vec{x}) = \frac{4G}{r} \Lambda_{ij,kl}(\hat{n}) \int d^3x' \, T_{kl}(t - r + x'\cdot\hat{n}, \vec{x}')}$$

we drop the bar as it has no trace

OR for Slow sources:

$$\boxed{h_{ij}^{TT}(t,\vec{x}) = \frac{4G}{r} \Lambda_{ij,ab}(\hat{n}) \sum_{l=0}^{\infty} \frac{\hat{n}_L}{l!} \int d^3x' \, \partial_t^l T^{ab}(t-r,\vec{x}') \, x'^L}$$

- Notice the even before switching to TT gauge, it is enough to focus on spatial part $\bar{h}_{ij}$. The rest will be fixed by Lorentz gauge conditions:

$$\partial_\alpha \bar{h}^{\alpha\mu} = 0 \quad \rightarrow \quad -i\omega \bar{h}^{0\mu} + \partial_i \bar{h}^{i\mu} = 0$$

$$\rightarrow \bar{h}^{0i} = \frac{1}{i\omega} \partial_j \bar{h}^{ji} \qquad \bar{h}^{00} = -\frac{1}{\omega^2} \partial_i \partial_j \bar{h}^{ij}$$

Then we can take the Fourier transform back.

▶ Focusing on Non-relativistic sources, it is possible to re-write the expressions in a more suggestive way:

$$h^{TT}_{ij}(t,\vec{x}) = \frac{4G}{r}\Lambda_{ij,kl}(\hat{n})\left[S^{kl}(t-r) + \hat{n}_a \dot{S}^{kl,a}(t-r) + \dots\right]$$

in which
$$S^{ij}_{(t)} \equiv \int d^3x'\, T^{ij}(t,\vec{x}')$$
$$S^{ij,k} \equiv \int d^3x'\, T^{ij}(t,\vec{x}')\, x'^k \quad , \text{etc.}$$

moments of the stress tensor

spatial part of $T^{\mu\nu}$

- We can define moments of T^{00} and T^{0i}

$$M = \int d^3x\, T^{00} \qquad M^i = \int d^3x\, T^{00} x^i \qquad \overbrace{M^{ij}}^{I^{ij}} = \int d^3x\, T^{00} x^i x^j \quad \text{etc.}$$

$$P^i = \int d^3x\, T^{0i} \qquad P^{i,j} = \int d^3x\, T^{0i} x^j \qquad \text{etc.}$$

These are related by energy-momentum conservation:

- $\dot{M} = 0$, $\dot{P}^i = 0$ standard argument.

$$\dot{M} = \int d^3x\, \partial_t T^{00} = -\int d^3x\, \partial_i T^{i0} = 0$$

$$\dot{P}^i = \int d^3x\, \partial_t T^{0i} = -\int d^3x\, \partial_j T^{ji} = 0$$

- $$\dot{M}^i = \int d^3x\, \partial_t T^{00} x^i = -\int d^3x\, \partial_j T^{j0} x^i = \int d^3x\, T^{i0} = P^i$$

$\partial_t (M R^i)$

again standard.
Center of mass moves with total momentum.

$$\dot{P}^{i,j} = \int d^3x\, \partial_t T^{0i} x^j = -\int d^3x\, \partial_k T^{ki} x^j = \int d^3x\, T^{ij} = \boxed{S^{ij}}$$

- $$\dot{M}^{ij} \equiv \dot{I}^{ij} = \int d^3x\, \partial_t T^{00} x^i x^j = -\int d^3x\, \partial_k T^{k0} x^i x^j = \boxed{P^{i,j} + P^{j,i}}$$

$$\Longrightarrow \boxed{S^{ij} = \frac{1}{2} \ddot{I}^{ij}}$$

$\int d^3x\, T^{ij}$ $\int d^3x\, T^{00} x^i x^j$

HW: show that

$$\dot{S}^{ij,k} = \frac{1}{2} \dddot{M}^{ijk} + \frac{1}{3}\left(\ddot{P}^{i,jk} + \ddot{P}^{j,ik} - 2\ddot{P}^{k,ij}\right)$$

$$\Longrightarrow \quad h_{ij}^{TT}(t,\vec{x}) = \frac{2G}{r}\Lambda_{ij,kl}(\hat{n})\,\ddot{I}^{kl}(t-r) + \cdots$$

symmetric not trace-less. $I^{kl} = Q^{kl} + \frac{1}{3}\delta^{kl} I$

traceless.

$$= \frac{2G}{r}\underline{\Lambda_{ij,kl}(\hat{n})}\,\ddot{Q}^{kl}(t-r)$$

$$= \boxed{\frac{2G}{r}\ddot{Q}^{TT}_{ij}(t-r) + \cdots}$$

This is known as <u>quadrapole formula</u>.

- We can explicitly construct the expression for h_+ and $h_\times$ as follows:

1. first we take $\hat{n} = \hat{z}$: $\longrightarrow \pi = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$

$$\Lambda_{ij,kl}A^{kl} = \left(\pi_{ik}\pi_{jl} - \frac{1}{2}\pi_{ij}\pi_{kl}\right)A^{kl}$$

$$= \pi\cdot A\cdot\pi - \frac{1}{2}\pi\,\mathrm{tr}(\pi\cdot A)$$

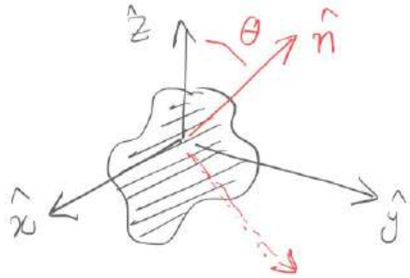
$$= \begin{bmatrix} (A_{11}-A_{22})/2 & A_{12} & 0 \\ \sim & \sim & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then for $\hat{n} = \hat{z}$:

$$\boxed{\begin{aligned} h_+(t,\vec{x}) &= \frac{G}{r}\left(\ddot{Q}_{11} - \ddot{Q}_{22}\right)(t-r) \\ h_\times(t,\vec{x}) &= \frac{2G}{r}\ddot{Q}_{12}(t-r) \end{aligned}}$$

2. Then for arbitrary $\hat{n}$, we just rotate Q_{ij} into a frame in which $\hat{n} \to \hat{z}$

for example $\hat{n} \sim (\theta, \varphi = \pi/2)$



we only require a rotation arround $\hat{x}$:

$$Q'_{ij} = R_{ik} R_{jl} Q_{kl} \qquad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$Q'_{11} = R_{1k} R_{1l} Q_{kl} = Q_{11}$$

$$Q'_{22} = R_{2k} R_{2l} Q_{kl} = \cos^2\theta \, Q_{22} + \sin^2\theta \, Q_{33} - 2\sin\theta\cos\theta \, Q_{23}$$

$$Q'_{12} = R_{1k} R_{2l} Q_{kl} = \cos\theta \, Q_{12} - \sin\theta \, Q_{13}$$

$$\Rightarrow \boxed{\begin{aligned} h_+ &= \frac{G}{r}\left[\ddot{Q}_{11} - \cos^2\theta \, \ddot{Q}_{22} - \sin^2\theta \, \ddot{Q}_{33} + \sin 2\theta \, \ddot{Q}_{23}\right] \\ h_\times &= \frac{2G}{r}\left[\cos\theta \, \ddot{Q}_{12} - \sin\theta \, \ddot{Q}_{13}\right] \end{aligned}}$$

HW : work out the more general form.

- We can calculate the total radiated power, by looking at the total energy flux :

$$P = \oint d^2x \; n_i t^{i0}$$

this is the total power loss :

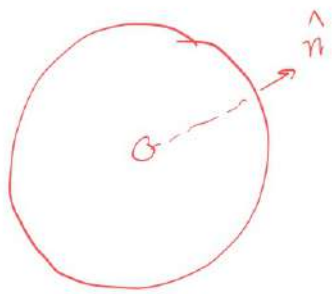
$$\frac{dE}{dt} = \int d^3x \; \partial_t (t + T)^{00}$$

$$= -\int d^3x \; \partial_i (t + T)^{i0}$$

$$= -\oint d^2x \; n_i (t + \cancel{T})^{i0}$$

no matter at infinity.

Notice that the surface integral is not zero since the fields decay as $\left(\frac{1}{r}\right)^2$ which will be compensated by the growth of the surface $\sim r^2$ and gives finite answer.



$$t^{i0} = \frac{1}{32\pi G}\left\langle \partial^i h^{TT}_{ab}\, \dot{h}^{TT}_{ab} \right\rangle$$

with $h^{TT}_{ij} = \frac{2G}{r}\Lambda_{ij,kl}\,\ddot{Q}^{kl}(t-r)$. time derivative is simple

Spatial derivative only acts on Q in the end because OW creates more powers of $1/r^2$ ... which vanish:

$$P = -\int d\Omega\, r^2 \frac{1}{32\pi G}\left(\frac{2G}{r}\right)^2 \left\langle \dddot{Q}^{TT}_{ab}\, \dddot{Q}^{TT\,ab}(t-r) \right\rangle$$

$$= -\frac{G}{8\pi}\int d\Omega\, \underbrace{\Lambda_{ij,kl}\,\Lambda^{ij}{}_{mn}}_{=\Lambda_{kl,mn}} \left\langle \dddot{Q}^{kl}\, \dddot{Q}^{mn} \right\rangle$$

doesn't depend on direction
it only depends on $(t-r)$

So the angular integral only cares about Λ:

$\int d\Omega\, \Lambda_{kl,mn}$; this is a bunch of integrals

on $\int d\Omega\, n_i n_j = \frac{4\pi}{3}\delta_{ij}$

$\int d\Omega\, n_i n_j n_k n_l = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \ldots)$

structure is fixed
coeff is fixed by trace

$$\int d\Omega\, \Lambda_{kl,mn} = \frac{2\pi}{15}\left(11\,\delta_{km}\delta_{ln} - \underbrace{4\,\delta_{kl}\delta_{mn}}_{\substack{\text{no cont.} \\ Q \text{ is traceless}}} + \delta_{kn}\delta_{lm}\right)$$

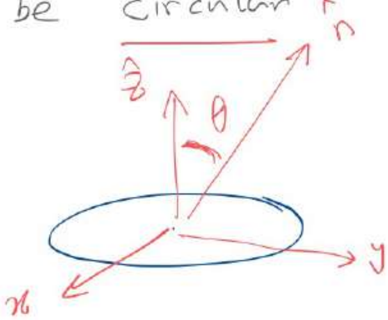
Plugging back we obtain:

$$\boxed{P = -\frac{G}{5}\left\langle \dddot{Q}_{ij}\,\dddot{Q}^{ij}\right\rangle}$$

total energy loss
per unit time.

⑦ Hulse - Tylor binary

Consider a circular binary for simplicity. At this order we can take the dynamics to be Newtonian. [You can consider the force between them not to be gravitational for now!] The problem is reduced to a single particle of the reduced mass $\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$, in an external potential given by the distance between them $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$. We take the orbit to be circular and in x-y plane:



$$x(t) = R\, \mathrm{Cos}(\omega_s t)$$
$$y(t) = R\, \mathrm{Sin}(\omega_s t)$$

$$I_{ij} = \int d^3x \; \delta(x - x(t))\, \delta(y - y(t))\, \delta(z)\, \mu\, x^i x^j$$

$$\rightarrow \quad I_{i3} = 0 \qquad I_{11} = \mu R^2 \mathrm{Cos}^2(\omega_s t) \qquad I_{12} = \mu R^2 \mathrm{Cos}(\omega_s t)\, \mathrm{Sin}(\omega_s t)$$
$$I_{22} = \mu R^2 \mathrm{Sin}^2(\omega_s t)$$

$$\rightarrow \quad \ddot{I}_{11} - \ddot{I}_{22} = 2\mu R^2 \omega_s^2 \mathrm{Cos}(2\omega_s t) \qquad \ddot{I}_{12} = 2\mu R^2 \omega_s^2 \mathrm{Sin}(2\omega_s t)$$

Next we should rotate to the direction of propagation and make it traceless:

$$\boxed{\begin{aligned} h_+(t,\vec{x}) &= \frac{4G\mu R^2\omega_s^2}{r}\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\omega_s(t-r)\right) \\ h_\times(t,\vec{x}) &= \frac{4G\mu R^2\omega_s^2}{r}\left(\cos\theta\right)\sin\left(2\omega_s(t-r)\right) \end{aligned}}$$

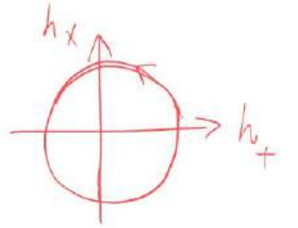
- Notice that since the system is cylyderically symmetric, this is the result for generic φ.

- For a source frequency of ω_s, GW frequency will be $2\omega_s$. This is because we are looking at quadrupole approximation $x^2 \sim \cos(\omega_s t)^2 \sim \cos(2\omega_s t)$.

- The amplitude of h_+ and $h_\times$ depends on the direction of propagation:

  E.g. for edge-on $(\theta = \pi/2)$: $h_\times = 0$ so it is linearly polarized. Notice that $h_+ \neq 0$ for all θ.

  E.g. for head-on $(\theta = 0)$: $h_+ = h_\times$ with $\pi/2$ phase difference. So it will be circularly polarized



For other values of θ this will be elliptic polarization

- We can calculate the radiated power per angle :

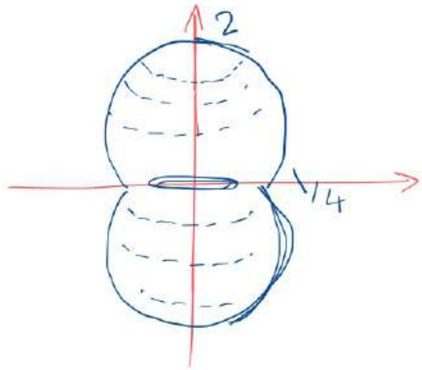
$$\frac{dP}{d\Omega} = r^2 t_{0i} n^i$$

$$= \frac{r^2}{32\pi G} \left\langle |h_+|^2 + |h_\times|^2 \right\rangle (2\omega_s)^2$$

$$= \frac{r^2}{32\pi G} \frac{(4G\mu R^2 \omega_s^2)^2}{r^2} 4\omega_s^2 \left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta \right] \left(\frac{1}{2}+\frac{1}{2}\right)$$

$$= \boxed{\frac{2G\mu^2 R^4 \omega_s^6}{\pi} \left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta \right]}$$

This has an angular dependence :
(like peanuts)



Total radiate power will be

$$P = \frac{32}{5} G\mu^2 R^4 \omega_s^6 \longrightarrow \Delta E = P\left(\frac{2\pi}{\omega_s}\right) = \left(\frac{64\pi}{5}\right) G\mu^2 R^4 \omega_s^5$$

energy loss per cycle.

$$\frac{\Delta E}{E_{kin}} = \left(\frac{128\pi}{5}\right) \underbrace{G\mu R^2 \omega_s^3}_{\left(\frac{G\mu}{R}\right) v^3}$$

$E_{kin} = 1/2\, \mu\, (\omega_s R)^2$

- For Hydrogen atom

$$\mu \simeq m_e \ , \quad R \simeq a_B \sim \frac{1}{m_e \alpha} \ , \quad v \sim \alpha$$

$$\frac{\Delta E}{E} \sim \alpha \left(\frac{l_P}{a_B} \right)^2 \sim 10^{-2} \left(\frac{10^{-35}\,\mathrm{cm}}{10^{-9}\,\mathrm{cm}} \right)^2 \sim 10^{-54} \ !$$

- For a Earth-Sun : Kepler's law : $\omega_s^2 = \frac{GM}{R^3}$ total mass $(m_1 + m_2)$

$m_1 \gg m_2$: $\mu \sim m_2$, $M \sim m_1$

$$\frac{\Delta E}{E} \sim O(100) \ G m_2 R^2 \left(\frac{G m_1}{R^3} \right)^{3/2} \sim O(100) \left(\frac{G m_1}{R} \right)^{5/2} \left(\frac{m_2}{m_1} \right)$$

$$\sim 10^2 \left(\frac{3\,\mathrm{km}}{10^8\,\mathrm{km}} \right)^{5/2} \left(\frac{10^{24}\,\mathrm{kg}}{10^{30}\,\mathrm{kg}} \right) \sim 10^{-22} \ !$$

- For the same masses $m = m_1 = m_2$: $\omega_s^2 = \frac{2Gm}{R^3}$, $\mu = m/2$

$$\frac{\Delta E}{E} \sim 10^2 \ G \frac{m}{2} R^2 \left(\frac{2Gm}{R^3} \right)^{3/2} \sim 10^2 \left(\frac{Gm}{R} \right)^{5/2} \ll 1$$

Therefore the assumption of circular orbit, which is quasi-stationary is a good assumption.
In the end, for observations we must take into account these effects for data analysis.

- We calculate the rate the size of the binary shrinks:

$$E_{bin} = \frac{-G m_1 m_2}{2R} \qquad P = -\frac{dE_{bin}}{dt}$$

$$-\frac{G m_1 m_2}{2R^2}\dot{R} = \frac{32}{5} G \mu^2 R^4 \omega_s^6 \implies \boxed{\dot{R} = -\frac{8}{5}\frac{(2GM)^3}{R^3}\left(\frac{\mu}{M}\right)}$$

time to collapse $\quad \tau \sim \dfrac{R_0^4}{(2GM)^3 (\mu/M)}$

- As the radius shrinks, the two bodies move arround each other faster, and therefore the frequency of GW emitted increases.

$$2\dot{\omega}_s \omega_s = -\frac{3GM}{R^4}\dot{R} \qquad \text{then we use Kepler: } R = \left(\frac{GM}{\omega_s^2}\right)^{1/3}$$

We obtain, re-writing in terms of $\omega_{GW} = 2\omega_s$

$$\frac{\dot{\omega}_{GW}}{\omega_{GW}^2} = \left(\frac{2^{11/3}}{5}\right)(2GM_c)^{5/3}\omega_{GW}^{5/3}$$

where we have defined Chirp mass : $M_c \equiv M^{2/5}\mu^{3/5} = \dfrac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$

We can integrate this equation to obtain $\omega_{GW}(t)$ ($\int_t^{t_{col}}$) :

$$\omega_{GW}(t) \simeq (2GM_c)^{-5/8}(t_{col}-t)^{-3/8}$$

This means that we can approximately look at the wave form :

$$h \sim \omega_{GW}^{2/3} \cos\left(-\int_t^{t_{col}} \omega_{GW}(t')dt'\right)$$

So we have $\left(R_c \equiv 2GM_c\right)$

$$h \sim \# \left[R_c(t_{col}-t)\right]^{-1/4} \cos\left(\# \left[R_c\left(t_{col}-t\right)\right]^{5/8}\right)$$