Review of GR

① EP

The principle of the equivalence of Gravity and Inertia tells us how an arbitrary system responds to an external gravitational field.

Weak EP: the fact the inertial (m_i) and gravitational (m_g) mass are equal (Galileo)

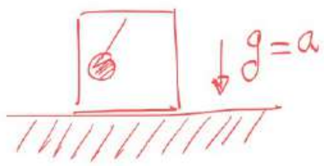
$\Rightarrow$ all particles accelerate the same in a given external gravitational field.

$$\cancel{m_i}\, \ddot{\vec{X}} = -\cancel{m_g}\, \vec{\nabla}\phi$$

$\Rightarrow$ we cannot distinguish

1) 

from

2) 

by watching particle trajectories [assuming a small box]

$$m_i \ddot{\vec{X}} = -m_i \vec{a} + \vec{F}_{int} \qquad\qquad m_i \ddot{\vec{X}} = m_g \vec{g} + \vec{F}_{int}$$

$\Rightarrow$ We can remove constant gravitational field by coordinate transformation

$$\vec{X} = \vec{X}' + \frac{1}{2}\vec{g}\,t^2 \quad \Rightarrow \quad m\,\ddot{\vec{X}}' = \vec{F}_{int}$$

• Notice that this is the free-fall frame $\Rightarrow$ No gravity

$\boxed{\text{Strong EP}}$ The above cancellation happens for all physical systems, (not just particle mechanics)

• locally, it is impossible to detect the effect of gravity $\Longrightarrow$ Special Relativity

▶ We can argue, using thought experiments, that gravity implies curved space-time.

1) Bending of light

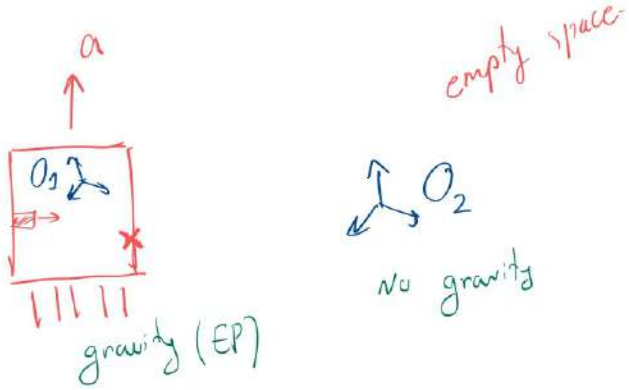
– accelerated experiment

O_2: light moves straight line

box has gone up

hitting spot is lower.

no gravity.

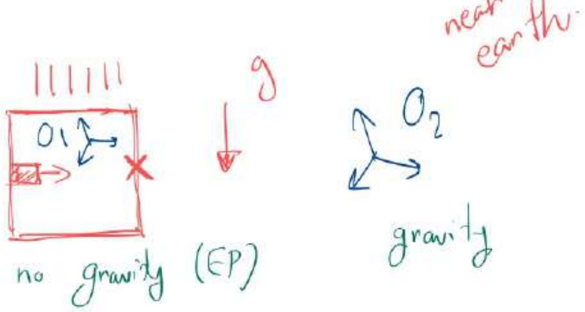


O_1: feels gravity (according to EP)

light trajectory is bent

hitting spot is lower

– Free fall experiment



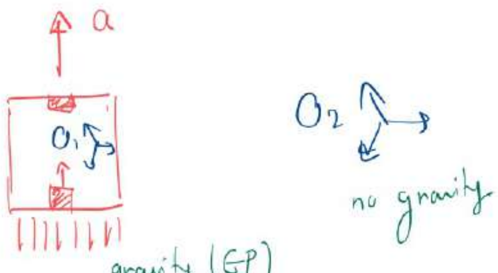
O_1: free fall means no gravity
light moves in straight line.
hits in front.

O_2: the box is going down, so the spot
must be higher $\longrightarrow$ so light trajectory
must be bent downwards.

$\Longrightarrow$ Whenever there is gravity (or inertial force) light
bends $\Longrightarrow$ Interpreted as $\boxed{\text{Curved Space}}$

2) Frequency Shift

- accelerated experiment



O_2: detector moving away from
the source with higher velocity
Doppler red shift.

O_1: light climbing up gravity gets red-shifted.

- free fall experiment

O_1: no redshift at all

O_2: the detector moves toward source. The blueshift
must be cancelled by the gravitational red-shift of
light.

$\Rightarrow$ whenever there is gravity (or inertial force) we see frequency shift. $\Rightarrow$ Interpreted as $\boxed{\text{Curving "time".}}$

(2) More mathematics

- Single particle under the influence of gravity

Choose free fall coordinate system to remove gravity (locally)

Special Relativity

$$\frac{d^2\xi^\mu}{d\tau^2} = 0 \qquad d\tau^2 = -\eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$$

proper time

$$\eta = \begin{pmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & +1 \end{pmatrix}$$

- Re-do the coordinate transformation to a generic coordinate

$$\xi^\mu \longrightarrow x^\mu \quad : \quad \xi^\mu(x^\alpha(\tau))$$

$$0 = \frac{d}{d\tau}\left(\frac{\partial \xi^\mu}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}\right) = \frac{\partial \xi^\mu}{\partial x^\alpha}\frac{d^2x^\alpha}{d\tau^2} + \frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

$$\rightarrow \boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0}$$

gravitational force as a result of coordinate transformation.

"Connection"

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial x^\mu}{\partial \xi^\lambda}\frac{\partial^2 \xi^\lambda}{\partial x^\alpha \partial x^\beta}$$

for the proper time

$$\boxed{d\tau^2 = -g_{\mu\nu}\, dx^\mu\, dx^\nu}$$

"metric"

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}$$

- For massless particles we can use another prameter instead of proper time which is zero ($0 = -g_{\mu\nu}\, dx^\mu dx^\nu$)
- From the above relations one can write Γ in terms of the derivative of the metric

$$\boxed{\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda}\left[\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}\right]}$$

- The equation for the particle trajectory is a "geodesic" (extremal path between two points) with respect to the metric:

$$\tau \equiv \int_A^B d\tau = \int_A^B d\lambda \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}$$

$x^\alpha \to x^\alpha + \delta x^\alpha$, demanding $\delta\tau = 0$ gives the geodesic equation.

(3) More Systematic

It is annoying to start from equations of special relativity every time, and perform a general coordinate transformation (to describe gravity).

Instead we identify different class of objects according to their transformation

▶ Tensors (many quantities are in this category)

$$T^{\mu}{}_{\nu}{}^{\rho}(x) \qquad\qquad x^{\mu} \to x'^{\mu}$$

$$T'^{\mu}{}_{\nu}{}^{\rho}(x') = \left(\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\right)\left(\frac{\partial x^{\beta}}{\partial x'^{\nu}}\right)\left(\frac{\partial x'^{\rho}}{\partial x^{\gamma}}\right) T^{\alpha}{}_{\beta}{}^{\gamma}(x)$$

e.g. metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ (defined $g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}{}_{\nu}$) are tensor (prove)

- Tensorial equation will have the same form in any coordinates : $A^{\alpha\beta} = B^{\alpha\beta} \longrightarrow A'^{\alpha\beta} = B'^{\alpha\beta}$
- If a tensor vanishes in a specific coordinate system, it vanishes in all coordinates.

- We can manipulate tensors to obtain other tensors : linear combination, product, contraction (raising / lowering)

Not all interesting quantities are tensors :

- E.g. Connection is <u>not</u> :

$$\Gamma'^{\mu}_{\alpha\beta} = \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial x^{\rho}}{\partial x'^{\alpha}} \frac{\partial x^{\sigma}}{\partial x'^{\beta}} \Gamma^{\lambda}_{\rho\sigma} + \boxed{\frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial^2 x^{\lambda}}{\partial x'^{\alpha} \partial x'^{\beta}}}$$

(prove this from the definition)

We expect this : We know that in the free fall frame $\Gamma \equiv 0$ if it was a tensor, it would remain zero in all frames.

- Partial derivative of a tensor is <u>not</u> :

$$V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} V^{\alpha}(x) \xrightarrow{\frac{\partial}{\partial x'^{\nu}}}$$

$$\underbrace{\partial'_{\nu}}_{\frac{\partial}{\partial x'^{\nu}}} V'^{\mu} = \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \underbrace{\partial_{\beta}}_{\frac{\partial}{\partial x^{\beta}}} V^{\alpha} + \boxed{\frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}}} V^{\alpha}$$

We expect this also : these are components of a vector field in some non-Cartesian coordinate system. We must take derivative of the unit vectors as well.

Covariant Derivative

$$\boxed{\nabla_\nu V^\mu \equiv \partial_\nu V^\mu + \Gamma^\mu_{\nu\alpha} V^\alpha}$$

this object transforms as a tensor!

Can be generalized to any tensor field.

- This is a differential operator: - linear - Leibniz rule.
- On scalars reduce to partial derivatives
- It is metric compatible: $\nabla_\alpha g_{\mu\nu} = 0$

- Volume element in integration (important because we like actions)

$$d^4x' = \underbrace{\left\| \frac{\partial x'}{\partial x} \right\|}_{\text{absolute value of the Jacobian of the transformation}} d^4x$$

An important combination: $\boxed{\sqrt{-g}\, d^4x}$

g → determinant of metric (as a matrix)

- It is a scalar

$$\sqrt{-g'}\, d^4x' = \sqrt{-g}\, d^4x$$

Trick: Write equations of special relativity such that they take the same form in any coordinate system, using tensors, covariant derivative, etc.

$\Rightarrow$ According to EP these are the equations in the presence of an external gravitational field.

Def: An equation with the above property is called <u>Generally Covariant.</u>

(1) Free Particle again

Special relativity: $\underbrace{\frac{d}{d\tau}}_{u^\alpha \partial_\alpha} u^\mu = 0 \quad , \quad \eta_{\mu\nu} u^\mu u^\nu = -1$

make it generally covariant: $\quad u^\alpha \nabla_\alpha u^\mu = 0 \quad , \quad g_{\mu\nu} u^\mu u^\nu = -1$

(2) Lagrangian

Lagrangian density

Minkowski metric

$$\underbrace{S}_{\text{action}} = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

ϕ: scalar field [it could be vector field, etc.]

make it form invariant

$\eta \to g$

$$S = \int (d^4x \sqrt{-g}) \, \mathcal{L}(\phi, \nabla_\mu \phi)$$

scalar (rank 0 tensor)

cov. deri

(2.1) Klein-Gordon:

$$S = \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \right]$$

$$\text{e.o.m} \longrightarrow \quad -\underbrace{\eta^{\mu\nu}\partial_\mu\partial_\nu}_{\partial^2}\phi - m^2\phi = 0$$

With gravity:

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\,\nabla_\nu\phi - \frac{1}{2}m^2\phi^2 \right]$$

same as ∂_μ

$$\text{e.o.m} \longrightarrow \quad \boxed{-g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - m^2\phi = 0}$$

$$\longrightarrow -\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right)$$

Notice: $\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,V^\mu)$

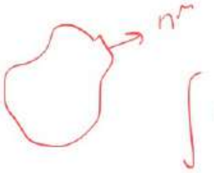
so divergence theorem is as usual with $\sqrt{-g}$

$$\int d^4x\sqrt{-g}\,\nabla_\mu V^\mu = \int d^4x\,\partial_\mu(\sqrt{-g}\,V^\mu) \quad \text{total derivative}$$

(2.2) Conserved Currents

Noether's theorem: Global Symmetries of action $\longrightarrow$ Conservation law

e.g. $U(1)$ symm $\longrightarrow \partial_\mu J^\mu = 0 \quad \Rightarrow \quad \nabla_\mu J^\mu = 0$



$$\int n_\mu J^\mu = 0 \qquad\qquad \int n_\mu\sqrt{-g}\,J^\mu = 0$$

space-time translation symm $\longrightarrow \partial_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \nabla_\mu T^{\mu\nu} = 0$

However : $\nabla_\mu T^{\mu\nu} = 0 \rightarrow \boxed{\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, T^{\mu\nu}) = \underbrace{-\Gamma^\nu_{\alpha\beta}T^{\alpha\beta}}}$

gravitational force density changes the energy of the system.

③ (Perfect) fluid

$$T^{\mu\nu} = (\underset{\text{energy density}}{\rho} + \underset{\text{pressure}}{p})\, u^\mu u^\nu + p\,\eta^{\mu\nu} \quad \text{with} \quad u^\mu u^\nu \eta_{\mu\nu} = -1$$

in the rest frame of one element : $u^\mu = (1, \vec{0})$

$$T^{\mu\nu} \stackrel{*}{=} \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix} \qquad \text{with} \quad p(\rho)$$

Hydro equations are simply : $\partial_\mu T^{\mu\nu} = 0$

Make it generally covariant : $T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p\, g^{\mu\nu}$

$$u^\mu u^\nu g_{\mu\nu} = -1 \quad , \quad \nabla_\mu T^{\mu\nu} = 0$$

- In all of the examples above the form of the equations remain the same if we change coordinates : $x^\mu \rightarrow x'^\mu$

4 Equation for gravity

GR was born when Einstein wanted to write a Lorentz invariant equation describing gravitation. So the above logic doesn't simply apply here.

We can try to see how such equation would look like by looking at Newtonian gravity:

$$\nabla^2 \phi = 4\pi G \rho \rightarrow \text{mass density } (\sim \text{energy})$$

Newtonian potential

Newton's Constant

RHS: $\rho \longrightarrow T^{00}$ so in the final equation the RHS must be proportional to $T^{\mu\nu}$

1) Symmetric 2) Covariantly Conserved.

LHS: from the geodesic equation in the Newtonian limit we know that $g^{00} \simeq -(1+2\phi)$

So on the LHS must be a tensor which is built out of the 2nd derivative of the metric.

4.1 Curvature

Such a tensor exists and is known as the curvature tensor: The easiest way to define it is as follows:

$$[\nabla_\mu, \nabla_\nu] V^\rho \equiv \underline{\underline{R^\rho{}_{\sigma\mu\nu}}} V^\sigma$$

Riemann tensor

using def. we obtain

$$\boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}}$$

- from the above formula, the fact that it is a tensor is not obvious, but all non-tensorial parts cancel.
- It involves $\partial\Gamma \sim \partial\partial g$ as advertised.
- It has a bunch of properties $\longrightarrow$ #20

  - $R_{\rho\sigma\mu\nu}$ anti-sym anti-sym
  - $R_{(\rho\sigma)(\mu\nu)}$ Symm.
  - $R_{\rho[\sigma\mu\nu]} = 0$

- Traces: $\underbrace{R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu}}_{\text{Ricci tensor}}$ #10 $\qquad \underbrace{R \equiv R^{\alpha\beta}{}_{\alpha\beta}}_{\text{Ricci scalar}}$

- Decomposition: $R_{\rho\sigma\mu\nu} = \left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho} \right) - \frac{1}{3} g_{\rho[\mu} g_{\nu]\sigma} R$

$$+ C_{\rho\sigma\mu\nu} \rightarrow \text{Weyl tensor}$$

#10

- Bianchi Identity : $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

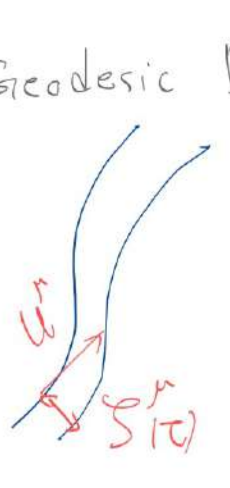
$$\xrightarrow[\text{twice}]{\text{Contraction}} \quad \nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0$$

- Why this is called Curvature tensor ?
  - Spacetime is flat, i.e. Minkowski metric perhaps written in a different coordinate, iff Riemann tensor vanishes everywhere.

Notice : By EP, in every spacetime, even when Riem $\neq 0$, we can find a free fall coordinate for which the metric will be η. But this is only locally. If we find a coordinate that the metric becomes η everywhere, then it is flat spacetime and Riem $= 0$. Inversely if Riem $= 0$, then we can always find such coord.

  - Geodesic Deviation

(ex: prove this)



$$\boxed{U^\alpha \nabla_\alpha \left(U^\beta \nabla_\beta \xi^\mu \right) = R^\mu{}_{\nu\rho\sigma} U^\nu U^\rho \xi^\sigma}$$

$\frac{D^2 \xi^\mu}{D\tau^2}$ acceleration of the deviation vector

Curvature causes deviation of geodesics.

(4.2) Einstein Equations

$$\boxed{}_{\mu\nu} = \kappa\, T_{\mu\nu}$$

- tensor, symmetric in $\mu\nu$, 2nd derivative in metric
Covariantly conserved $D_\mu T^{\mu\nu} = 0$, reduce to Newtonian gravity

Einstein tensor.

$$\boxed{G_{\mu\nu} \equiv \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 8\pi G\, T_{\mu\nu}}$$

$T \equiv T^{\alpha}{}_{\alpha}$

- equivalent form : $R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$

- Notice : $R = -8\pi G T$

- Only trace part of curvature is given by $T_{\mu\nu}$: Weyl tensor, due to Bianchi identity, is related to $T_{\mu\nu}$ through a diff eq.
  ↳ We can have curvature without having matter @ that point.

- Lagrangian formulation

RHS is given by variation of

$$S = \int \left(d^4x \sqrt{-g}\right) (R) \qquad \text{with respect to metric}$$

↳ satisfies general covariance.

$g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$
$S \to S + \delta S$ $\Rightarrow$ $\delta S = \int d^4x \sqrt{-g}\, G_{\mu\nu}\, \delta g^{\mu\nu}$

LHS : we "define" the energy-momentum tensor as variation of the action that describes matter (anything but gravity) with respect to the metric

$$S = \underbrace{\frac{1}{16\pi G}\int d^4x \sqrt{-g}\, R}_{S_{EH}} + S_M$$

$$\Longrightarrow \quad G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \checkmark$$

$$\boxed{T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}}$$

or equivalently

$$\delta S_M \equiv \frac{-1}{2}\int d^4x \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu}$$

- Why this is a good definition :
  - It is symmetric manifestly
  - It can be shown it is covariantly conserved : linearized coord. trans.

proof : By GC $\boxed{\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu}$: $\delta S_M = 0$

$$0 = \frac{-1}{2}\int d^4x \sqrt{-g}\, T^{\mu\nu} \left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\right)$$

$$= \frac{1}{2}\int d^4x \sqrt{-g} \left(\xi_\nu \nabla_\mu T^{\mu\nu} + \xi_\mu \nabla_\nu T^{\mu\nu}\right) = \int d^4x \sqrt{-g}\, \xi_\nu \left(\nabla_\mu T^{\mu\nu}\right)$$

- It may not coincide with Noether procedure but that is ambiguous → It can always be made symmetric then it coincides.

(4.3) Electromagnetic Analogy

general covariance $\sim$ gauge invariance

The way we built the theory of gravity is almost identical to the way we approach gauge theories, e.g. EM.

- We start from a theory which has $U(1)$ symmetry

$$\psi(x) \to e^{i\alpha} \psi(x) \qquad \forall \alpha$$

- We demand that it is invariant under local $U(1)$

$$\psi(x) \to e^{i\alpha(x)} \psi(x) \qquad \forall \alpha(x)$$

This is impossible because we always have derivatives, So we define covariant derivative

$$\text{(connection)} \quad D_\mu \psi \equiv \partial_\mu \psi - i A_\mu \psi$$

with a gauge field A_μ that transforms as: $A_\mu \to A_\mu + \partial_\mu \alpha$

then:
$$D_\mu \psi \longrightarrow e^{i\alpha(x)} D_\mu \psi$$

- Then the original Lagrangian will be gauge inv. $\mathcal{L}(\psi, D\psi)$

- To obtain an appropriate action for A_μ we notice

$$[D_\mu, D_\nu]\psi \equiv -i F_{\mu\nu} \psi$$

where we can explicitly see that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and it is gauge invariant.

- We can "define" the conserved current as $J^\mu \equiv \frac{\delta S_M}{\delta A_\mu}$ and it is easy to see that gauge invariance implies its conservation.

- Maxwell equations $\partial_\mu F^{\mu\nu} = J^\nu$, given the source and appropriate boundary conditions, do not give A_μ Completely: for any solution A_μ, $A'_\mu = A_\mu + \partial_\mu \alpha$ is as good!

- This is consistent with the fact that the 4 equations are not independent: $\partial_{\mu\nu} F^{\mu\nu} = 0$

- We fix the gauge freedom (redundancy) by putting an extra Constraint (gauge fixing): eg. $\partial_\mu A^\mu = 0$ Lorenz gauge

4.4 Coordinate Conditions

Einsten equations do not determine uniquely the metric.

For any solution $g_{\mu\nu}$: $g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$ is also a solution.

This is consistent with the fact that the equations are not independent: $\nabla_\mu G^{\mu\nu} = 0$