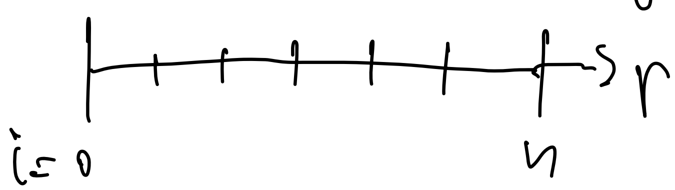
Computational astrophysics: numerical analysis

- In order to illustrate common issues in numerical analysis for (astro)physics, let us try to solve $\nabla^2\phi = 4\pi G\,\delta^{(3)}(\vec{r})$
- Sp. sym. [$G=1$]

$$\phi'' + \frac{2\phi'}{r} = 4\pi\,\delta^{(3)}(\vec{r}) \,, \quad \phi(\infty) = 0$$

From continuum to a grid



- Three types of numerical derivatives

$$\Delta_+ \phi(x) = \frac{\phi(x+h) - \phi(x)}{h}$$

$$\Delta_- \phi(x) = \frac{\phi(x) - \phi(x-h)}{h}$$

$$\delta\phi(x) = \frac{\phi(x+h) - \phi(x-h)}{2h}$$

→ how good are they?

error estimate $E = D_N - \phi'$, $D_N = \{\Delta_+, \Delta_-, \delta\}$

$$\phi(x \pm h) = \phi(x) \pm \phi' h + \frac{1}{2}\phi'' h^2 \pm \frac{1}{6}\phi''' h^3 + O(h^4)$$

$$E(\Delta_\pm) = \pm\frac{1}{2}\phi'' h + O(h^2)$$

$$E(\delta) = \frac{1}{6}\phi''' h^2 + O(h^3)$$

→ $E(h) \sim C h^{p}$ ⇝ order of accuracy

- H.O. derivatives

$$\delta^2 \phi = \delta\left(\frac{\phi(x+h) - \phi(x-h)}{2h}\right)$$

$$= \frac{\phi(x+2h) - \phi(x) - \phi(x) + \phi(x-2h)}{4h^2}$$

$$\overset{2h \to h}{=} \frac{\phi(x+h) - 2\phi(x) + \phi(x-h)}{h^2}$$

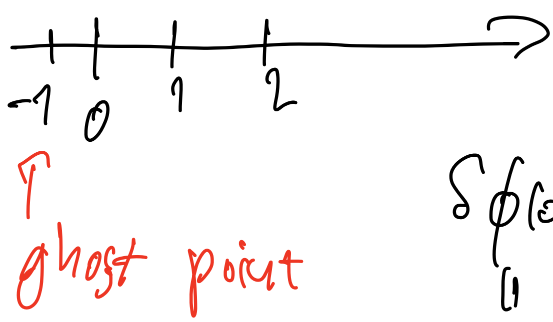
coordinate singularity $r=0$

$$\nabla^2\phi = \phi'' + \frac{2\phi'}{r}$$

- in order to have reg. $r\to 0$: [IVP approach]

$$\phi = c_0 + c_1 r + c_2 r^2 + \dots, \quad \frac{\phi'}{r} = \frac{c_1}{r} + 2c_2 + \dots \Rightarrow c_1 = 0$$

- $\lim_{r\to 0}\phi' = 0 \Rightarrow$ implement as a BC
- ghost point method



$$\delta\phi(0) = \frac{\phi(0+h) - \phi(0-h)}{2h} = 0$$

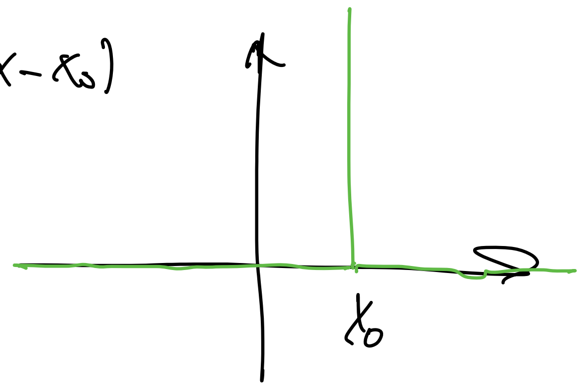
$$\Rightarrow \phi(-h) = \phi(h)$$

- regularize $\nabla^2\phi\big|_{r=0}$

as $\lim_{r\to 0}\frac{2\phi'}{r} = \phi''$ [L'Hopital]

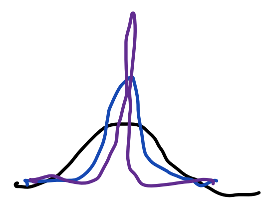
- Dirac delta source

$\delta(x-x_0)$



- as a sequence

e.g. $\lim_{\sigma\to 0}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$

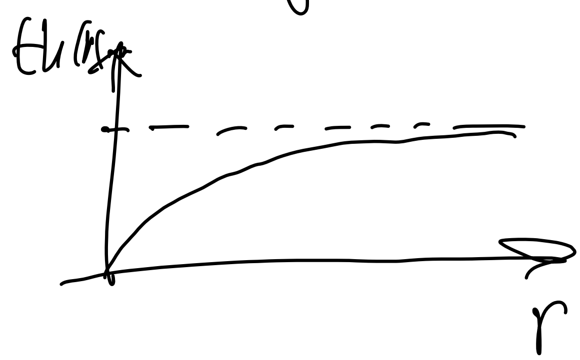


$\sigma_1>\sigma_2>\sigma_3$

- if we regularize Dirac δ source with a Gaussian all physical info must be taken from $r \gg 2\sigma$ region

(*) BC

- in general, if we know the asymptotic solution, we can impose BC at $\phi(r_p)$, $r < r_p$
- in this case, $\phi(r_p) = -M/r_p$
- we could also change coordinates to compactify the domain e.g. $\tanh(r)$



(*) How to test the numerics?

- comp. with the analytic solution in a domain where it exists
- Convergence test

$$E_h = \|\phi_h - \phi_*\| \qquad \|e\|_2 = \left[h \sum_{i=1}^{N} |e_i|^2 \right]^{\frac{1}{2}}$$

$$\phi_h \approx \phi_* + Ch^p$$

$\rightarrow$ take $\{\phi_{\frac{h}{4}}, \phi_{\frac{h}{2}}, \phi_h\}$

$$\phi_h - \phi_{\frac{h}{4}} = C\left[h^p - \left(\frac{h}{4}\right)^p\right]$$

$$\phi_{\frac{h}{2}} - \phi_{\frac{h}{4}} = C\left[\left(\frac{h}{2}\right)^p - \left(\frac{h}{4}\right)^p\right]$$

$$R \equiv \frac{\phi_h - \phi_{\frac{h}{4}}}{\phi_{\frac{h}{2}} - \phi_{\frac{h}{4}}} = \frac{4^p - 1}{2^p - 1} = 2^p + 1$$

$\downarrow p=2$

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