# PHYSICS of INFLATION - CLASSICAL DYNAMICS - Lecture 2

## Brief Recap

Inflation $\ddot{a} = 0 \longrightarrow \epsilon = -\frac{\dot{H}}{H^2} \ll 1$ and $|\eta| = \left|\frac{d\epsilon}{dt} \cdot \frac{1}{\epsilon H}\right| \ll 1$

↳ quasi dS spacetime. ↳ Inflation lasts enough time.

What kind of mechanism could give rise to inflation?

## 1. Vacuum energy as C.C.

Fill the universe with constant vacuum energy. It acts as a cosmological constant. E.E. gets modified:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = M_{pl}^{-2} T^{m}_{\mu\nu} - \Lambda g_{\mu\nu} \qquad M_{pl}^2 = \frac{1}{8\pi G}$$

Stress energy tensor associated to $\Lambda$: $T^{\Lambda}_{\mu\nu} = \frac{\Lambda}{M_{pl}^{-2}} g_{\mu\nu}$

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Compare with $T_{\mu\nu}$ of a perfect fluid with 4-velocity $u_\mu$

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$$

comoving fluid: $u_\mu = (-1, \vec{0}) \Rightarrow T_{\mu\nu} = \begin{cases} \rho & 00 \\ p a^2 & ii \end{cases}$

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$$\left.\begin{aligned} T^{\Lambda}_{00} &= + M_{pl}^2 \Lambda \\ T_{ii} &= - M_{pl}^2 \Lambda \end{aligned}\right\} \rightarrow$$

Vacuum energy behaves as a perfect fluid with $p = -\rho$! $(w = -1)$

During the expansion ordinary matter is redshifted away very fast so I end up in a perfect dS space.

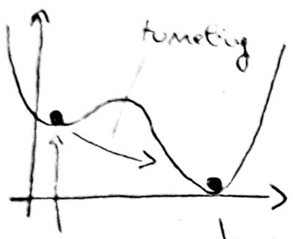
$\underline{\epsilon = \eta = 0}$

Problem: inflation never ends!

## 2. False vacuum inflation - Old inflation

Guth '81

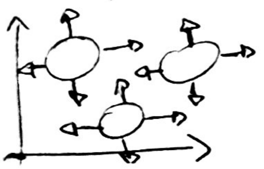
Consider a metastable vacuum:




Initially the universe is filled with vacuum energy in its false vacuum. The universe is inflating. Classically this configuration is stable, but quantum mechanically this configuration is not stable. ~~Thus~~ At a certain point the false vacuum will tunnel in the true one making inflation end.

~~Problem~~

End of inflation happens through bubble nucleation



$\Gamma$: decay rate: probability per unit time and volume

- $\Gamma \gg H_I^4$: bubble percolates and you can reach the new phase in a time $\ll H^{-1}$

  → No inflation

- $\Gamma \ll H_I^4$: Bubbles of the new phase cannot find each other. ~~Thus~~

  → No new phase

$$[\Gamma] = \frac{\cancel{*}}{[T][V]} = [E]^4$$

You can still tune $\Gamma$ in such a way that Bubbles meet but in this case the Universe is very inhomogeneous!

(~~But~~ Live inside 1 bubble
Universe homogeneous, energy given by scalar oscillation...
But $\Omega_k \sim$ curvature of Bubble → curvature dominated!)

3. Slow roll inflation (*)

Consider a scalar field minimally coupled to Einstein gravity:

$$S = \int d^4x \sqrt{-g}\left[\frac{M_{pl}^2}{2} R - \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right]$$

Its stress-energy tensor is:

$$T^\phi_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}}\left[\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\cdot\left(-\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right) - \frac{\sqrt{-g}}{2}\partial_\mu\phi\,\partial_\nu\phi\right]$$

$$= \partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}\left(-\frac{1}{2} g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - V(\phi)\right)$$

$$\left(\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu}\right)$$

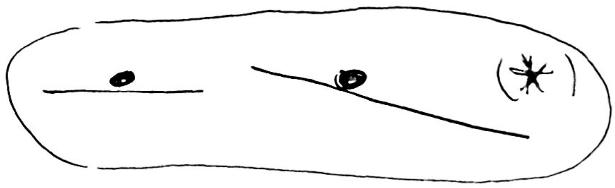
Focus on the homogeneous solution: $\phi(t)$. (We are assuming the field to be homogeneous and isotropic: $\phi(\vec{x},t) = \phi(t) + \delta\phi(\vec{x},t)$, $|\delta\phi(\vec{x},t)| \ll \phi(t)$.)

Then:

$$T^\phi_{\mu\nu} = \begin{cases} \dot{\phi}^2/2 + V(\phi) & 00 \\ -a^2\left(V(\phi) - \dot{\phi}^2/2\right) & ii \end{cases}$$

The scalar field behaves as a perfect fluid with $\rho = V + \frac{\dot{\phi}^2}{2}$ and $p = -V + \dot{\phi}^2/2$. Therefore we have inflation if

$$p/\rho = -1 \Rightarrow \frac{\dot{\phi}^2/2}{V} \ll 1$$ → The potential dominates the kinetic energy! (slow roll condition)



Dynamics of the scalar field

Calculate the EoM of $\phi(t)$

$$\partial_\mu\left(\frac{\delta S}{\delta \partial_\mu\phi}\right) - \frac{\delta S}{\delta\phi} = 0 \rightarrow \partial_\mu\left(-\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right) + \sqrt{-g}\, V'(\phi) = 0$$

$$-\left(\frac{\delta S}{\delta \dot\phi}\right)^{\cdot} - \frac{\delta S}{\delta \phi} = 0 \qquad S = \int d^4x \sqrt{-g}\left(\frac{\dot\phi^2}{2} - V(\phi)\right)$$

$$\frac{d}{dt}\left(\sqrt{-g}\,\dot\phi\right) + \sqrt{-g}\,V' = 0 \qquad \sqrt{-g} = a^3 \,, \quad V' \equiv \frac{\partial V}{\partial \phi}$$

$$3\dot a a^2 \dot\phi + \ddot\phi a^3 + a^3 V' = 0 \Big/ a^3$$

$$(*) \quad \boxed{\ddot\phi + 3H\dot\phi + V' = 0} \rightarrow \text{Klein Gordon equation in a curved background for a homogeneous field}$$

Hubble friction

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Notice that if we start in the situation in which $\dot\phi^2 \gg V$, then

$$\ddot\phi + 3H\dot\phi \simeq 0 \rightarrow \frac{\ddot\phi}{\dot\phi} = -3\frac{\dot a}{a} \,, \quad \dot\phi \propto a^{-3}$$

Now the energy of the field is $\rho \simeq \frac{\dot\phi^2}{2} \propto a^{-6}$

The kinetic phase is redshifted away! $\rightarrow$ The inflationary solution is an attractor!

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Then to have inflation I expect the friction term that dominates the kinetic term, i.e. $|\ddot\phi| \ll |\dot\phi|$

$$\Rightarrow \boxed{V' \simeq -3H\dot\phi} \quad (**)$$

<u>Slow roll parameters</u>

If the en. density of the scalar field permeates the universe:

$$H^2 = \frac{1}{3M_{pl}^2}\left(V + \frac{\dot\phi^2}{2}\right)$$

derive w.r.t. time

$$2H\dot H = \frac{1}{3M_{pl}^2}\left(V'\dot\phi + \dot\phi\ddot\phi\right) \overset{\text{Use } (*)}{=} \frac{\dot\phi}{3M_{pl}^2}\left(\cancel{V'} - 3H\dot\phi - \cancel{V'}\right) = \frac{-\cancel{3}\dot\phi^2 H}{\cancel{3}M_{pl}^2}$$

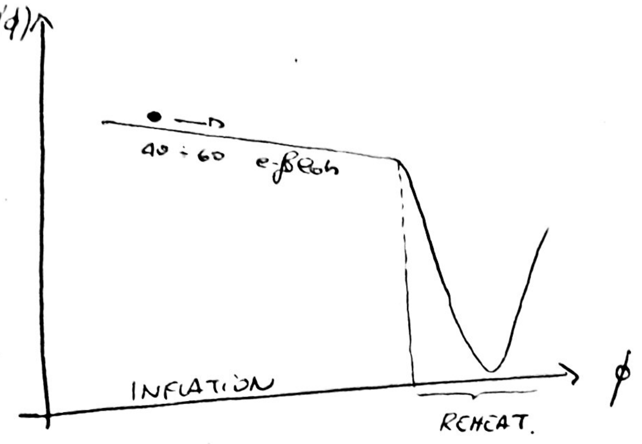
$$\Rightarrow \dot H = -\frac{1}{2M_{pl}^2}\dot\phi^2 \quad \leftarrow \text{no approx!}$$

Then $\epsilon$ is: $\quad \epsilon = -\dot{H}/H^2 = \dfrac{\dot{\phi}^2}{2M_{pl}^2 H^2}$

It can be also written in terms of the potential:

$$H^2 \simeq \frac{1}{3M_{pl}^2} V$$

$$V' \simeq -\frac{\dot{\phi}}{3H} \longrightarrow \underline{\epsilon \simeq \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2}$$




And what about $\eta$?

$$\eta = \frac{\dot{\epsilon}}{\epsilon H} = \underbrace{\left(\frac{\dot{\phi}\ddot{\phi}}{\cancel{M_{pl}^2} \cancel{H^2}} - \frac{\dot{\phi}^2}{\cancel{M_{pl}^2}}\frac{\dot{H}}{H^3}\right)}_{\dot{\epsilon}} \cdot \underbrace{\frac{2\cancel{M_{pl}^2}\cancel{H^2}}{\dot{\phi}^2}}_{\epsilon^{-1}} \cdot \frac{1}{H} =$$

$$= 2\frac{\ddot{\phi}}{\dot{\phi}H} - 2\frac{\dot{H}}{H^2} = 2\frac{\ddot{\phi}}{\dot{\phi}H} + 2\epsilon$$

Now using $V' = -3H\dot{\phi}$ we get:

$$\ddot{\phi} = -\frac{V''\dot{\phi}}{3H} + \frac{V'}{3}\frac{\dot{H}}{H^2}$$

then

$$\frac{2\ddot{\phi}}{\dot{\phi}H} = \frac{2}{3}\left(-\frac{V''}{H^2} + \frac{\dot{H}}{H^2}\underbrace{\frac{V'}{\dot{\phi}H}}_{-3}\right) = 2M_{pl}^2\frac{V''}{V} + 2\epsilon$$

$$\Rightarrow \eta = 4\epsilon - 2M_{pl}^2\frac{V''}{V}$$

Example : "Harmonic oscillator"

$$V(\phi) = \frac{1}{2} m^2 \phi^2$$

$$\begin{cases} \epsilon = \frac{M_{pl}^2}{2} \left( \frac{m^2 \phi}{\frac{1}{2} m^2 \phi^2} \right)^2 = 2 \frac{M_{pl}^2}{\phi^2} \\ \eta = 8 \frac{M_{pl}^2}{\phi^2} - 2 M_{pl}^2 \cdot \frac{\cancel{m^2}}{\frac{1}{2} \cancel{m^2} \phi^2} = 4 \frac{M_{pl}^2}{\phi^2} \end{cases}$$

To have inflation we need $\phi \gg M_{pl}$.

How much is the field excursion?

$$N \equiv \int_{a_i}^{a_f} d\ln a = \int_{t_i}^{t_f} H\, dt$$

$$H dt = \frac{H}{\dot{\phi}} d\phi \simeq - \frac{3H}{V'} H d\phi = \frac{1}{\sqrt{2\epsilon}} \frac{d\phi}{M_{pl}}$$

In our case: $\epsilon = 2 M_{pl}^2 / \phi^2 \rightarrow N = \int_{\phi_i}^{\phi_f} \frac{\phi}{2 M_{pl}} \frac{d\phi}{M_{pl}} = \frac{\phi_f^2 - \phi_i^2}{4 M_{pl}^2}$

Now $\phi_f = 0 \Rightarrow$ to have $40 \div 60$ e-folds $\phi_i \simeq (12 \div 15) M_{pl}$

Also the field excursion must be superplanckian!

Belongs to the class of Large field models of inflation.

<u>Phase space</u>

Consider now the EoM: $\left( \ddot{\phi} = \frac{d\dot{\phi}}{dt} = \frac{d\dot{\phi}}{d\phi} \cdot \dot{\phi} \right)$

$$\frac{d\dot{\phi}}{d\phi} \dot{\phi} + 3H\dot{\phi} = - m^2 \phi$$

$$H^2 = \frac{1}{3 M_{pl}^2} \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right)$$

$$\frac{d\dot{\phi}}{d\phi} = - \frac{\sqrt{\frac{3}{2}} \frac{\dot{\phi}}{M_{pl}} \left( \dot{\phi}^2 + m^2 \phi^2 \right)^{1/2} + m^2 \phi}{\dot{\phi}}$$

Focus now on $\phi \gg M_{pe}$

Kinetic domination: $\dot{\phi}^2 \gg m^2\phi^2$

EoM reduces to $\frac{d\dot{\phi}}{d\phi} = -\sqrt{\frac{3}{2}}\frac{|\dot{\phi}|}{M_{pe}}$, whose solution is $\dot{\phi} \propto e^{-\frac{\sqrt{3/2}}{M_{pe}}\phi}$

$\dot{\phi}$ redshift exponentially fast in the direction of motion

Attractor phase: $\dot{\phi}^2 \ll m^2\phi^2$

Look for fixed points: $\frac{d\dot{\phi}}{d\phi} = 0$

$\frac{3}{2}\frac{\phi^2}{M_{pe}^2}(\dot{\phi}^2 + m^2\phi^2) = m^4\phi^2$, whose solution is $\dot{\phi} = \pm\sqrt{\frac{2}{3}}\, m M_{pe}$

$\Rightarrow \dot{\phi}^2 = \frac{2}{3} m^2 M_{pe}^2 \ll m^2\phi^2 \rightarrow$ self consistent