Problem 1.
Start with the FRW metric in a flat universe. Change the time variable such that $dt = a(t)d\tau$. The variable τ is called "conformal" time. Can you explain why this change of coordinates is useful? Define a new Hubble parameter using derivatives with respect to conformal time $\mathcal{H} \equiv a'/a$, where $' = d/d\tau$. Write Friedmann equations in terms of conformal time and $\mathcal{H}$.

Problem 2. (For those who know GR)
Start with the geodesic equation for a massive particle

$$\frac{dU^\mu}{ds} + \Gamma^\mu_{\alpha\beta} U^\alpha U^\beta = 0 \tag{0.1}$$

where $U^\mu \equiv dX^\mu/ds$. Rewrite derivative with respect to arbitrary parameter s in terms of derivative with respect to the coordinates X^μ. Using this modified geodesic equation, find the equation for four-momentum P^μ in FRW cosmology and study its zero component. Derive the scaling of three-momentum $\boldsymbol{p}$ with the scale factor.

Problem 3.
Using the fact that for massive or massless particles the three-momentum $\boldsymbol{p}$ scales as $\boldsymbol{p} \sim 1/a(t)$, give an intuitive explanation for the scaling of energy density with $a(t)$ for a universe filled with matter and a universe filled with radiation. Using the continuity equation and definition $p = w\rho$, find the equation of state parameter w for matter and radiation. How the temperature of radiation scales with the scale factor? What would be w for a substance whose energy density is constant? Can you think of an example of such substance?

Problem 4.
Solve Friedmann equations for a fluid with generic w. How does the scale factor $a(t)$ scales with time in universes filled with matter, radiation and fluid with constant energy density? Find the scaling with conformal time. Draw a diagram $\log \rho$ vs $a(t)$ for different fluids. Which energy density dominates at early times and which one at late times?

Problem 5.
Solve Friedmann equations for a universe with two fluids: matter and radiation. Hint: use Friedmann equations with conformal time.

Problem 6.
Write down the action for a free massive scalar field minimally coupled to gravity. Calculate the stress-tensor for the scalar field. What is the equation of state parameter w for the scalar-field fluid in the limit $H \ll m$, where m is the mass of the scalar field? What happens if the field is in the minimum of the potential which is positive?

Problem 7.
Think about a curved universe. What would be the equation of state parameter for this universe? Can you find a parametric solution for a universe with matter and curvature?