Casual problems in Cosmology

Some problems to explore in detail most aspects of the Petnica lectures on Cosmology.

1. Homogeneous Cosmology

(a) Age of the Universe. Before the acceleration of the Universe was discovered in 1998, there were some indications against a matter dominated, flat universe. In particular, the age of globular clusters inferred from stellar evolution was larger than the age of the universe in such a model.
Suppose that we measure the age of a galaxy at redshift $z = 1$. How old would the galaxy have to be (at the time the light from it was emitted) in order to rule out the hypothesis that the Universe is matter dominated, i.e. $\Omega_m = 1$ and $\Omega_\Lambda = \Omega_K = \Omega_r = 0$? You can express the answer in terms of H_0, the Hubble parameter today.

(b) Einstein's static Universe. Originally, Einstein introduced the cosmological constant in his equations because it allowed for a static universe solution.
Find out the balance of matter content, the cosmological constant and the curvature of a static universe. What is its topology? Is it stable under perturbations?

(c) Milne Universe. Find the FRW solution for a open universe (that is, spatial curvature $k = -1$), in the limit of zero energy density. Show that the Ricci tensor is vanishing, and therefore the spacetime is actually Minkowski. Find the coordinate transformation that brings the solution to standard Minkowski form, and try to sketch the causal structure of this spacetime.
A nice discussion of this solution is in Mukhanov and Misner-Thorne-Wheeler.

(d) Cosmic string network. A network of cosmic strings has equation of state $w = -1/3$, and therefore its energy density will redshift as a^{-2}. How can we distinguish the presence of this exotic matter in a spatially flat universe from an open universe, where the spatial curvature term also redshifts as k/a^2?

(e) Lyttleton-Bondi Universe. Just so you understand how young is our understanding of the Universe (and of quantum field theory, which dictates the magnitudes of the charges of the proton and electron to be equal), here's something that Lyttleton and Bondi proposed in 1959.
Imagine that protons have a charge $e_p = (1 + y)e$, where e is the magnitude of the electron's charge, thus giving un-ionized hydrogen a net charge ye. Consider an astronomical volume of constant density containing just hydrogen. First, find the electric field at radius R and the value of y such that the electric repulsion felt by an atom is larger than the gravitational attraction (should you worry about matter external to the volume?). Then, show that the net force is proportional to the radius R, and therefore the radial velocity v of an atom is proportional to its distance R, assuming the density constant. (Of course, here there is a problem: the density dilutes. One should assume, as in the original paper, some sort of creation of matter to keep the density constant, but this would bring us too far). Finally, write the velocity in the form $v = R/T$. What is the physical meaning of T?

2. Newtonian approximation

(a) Fermi coordinates. Take the FRW metric, and make the explicit change of coordinates to the frame of a freely-falling observer. Calculate the lowest order deviations from the Minkowski metric, which are $O(x^2)$. From the point of view of us observers, for distances smaller than the horizon this describes a small, repulsive potential. At linear order, we can describe the gravitational attraction of our galaxy by just adding the Newtonian potential. In effect, we start from the Schwarzshild solution, go to Fermi coordinates, find the lowest-order terms and add them.
This is a way to estimate the smallness of the universe expansion on galactic scales. At what distance is the expansion comparable to the gravitational attraction of a galaxy ($M \simeq 10^8 M_\odot$)?

(b) McVittie solution. There is an exact solution of Einstein's equations describing a mass m embedded in an expanding universe. The metric is

$$\mathrm{d}s^2 = -\left(\frac{1-\mu}{1+\mu}\right)^2 \mathrm{d}t^2 + (1+\mu)^4 a(t)^2 \mathrm{d}\vec{x}^2 \,, \tag{1}$$

where $a(t)$ is the asymptotic cosmological scale factor, and $\mu = m/(2a(t)|\vec{x}|)$.
Show that this is an exact solution of Einstein's equations, provided that $a(t)$ solves the Friedmann equation $3M_{\mathrm{Pl}}^2 H(t)^2 = \rho(t)$. Then, linearize the solution for small mass: how do you interpret the resulting metric?

3. $F(R)$ gravity

Consider an action of the form

$$S = \int \mathrm{d}^4x \sqrt{-g} F(R) \,. \tag{2}$$

Let us verify that this is actually equivalent to General Relativity plus a scalar. First, show that the action above is the same as

$$S = \int \mathrm{d}^4x \sqrt{-g} [F'(A)(R-A) + F(A)] \tag{3}$$

where A is an auxiliary scalar field. At this point, we can use a conformal transformation of the metric to separate the Einstein-Hilbert action from that of a scalar field. The final result will be

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[R - \frac{3}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \,, \tag{4}$$

where

$$V(\phi) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \,, \qquad \phi = -\ln F'(A) \,. \tag{5}$$

What is happening to the scalar degree of freedom in the case of $F(R) = R$?

4. Cosmic Microwave Background

A small intro — Before reshift $z \simeq 1100$, photons, electrons and nuclei interact strongly, and can be approximately described by a relativistic perfect fluid, the baryon-photon plasma. Thermodynamic equilibrium requires this plasma to be a blackbody to an excellent approximation, as experimentally confirmed by the COBE mission. At $z \simeq 1100$, atoms (hydrogen and helium) form, a process traditionally called *recombination*, although it is the first time in the history of the Universe that electrons and nucleons combine into neutral atoms. After this happens, photons are free to stream to us today, and retain a Planckian spectrum. Because of the redshift, the radiation today is peaked at microwave frequencies, with a temperature of 2.7 K.

(a) After recombination, photons do not interact. Why is the form of the spectrum preserved today, and how does it evolve with time?
Given that we measure the temperature today to be 2.7 K, what was the temperature at recombination?

(b) The temperature of the blackbody spectrum is the *typical* energy scale of the photons. Compare this temperature to the ground state energy of the hydrogen atom, which is 13.6eV. Can you explain why it is quite lower?

(c) Should we worry that the CMB radiation cooks us, as in a giant microwave oven? Let's try to estimate its power in the following way. Suppose we want to prepare some tea, heating a mug of water initially at $T = 20°C$. A microwave oven will make it boil in a few minutes. If we leave the mug on the table, and consider only the effects of the CMB, what would the temperature increase be after 10 minutes?

5. Inflation

(a) During inflation, any light field will generate classical perturbations from quantum fluctuations. We have seen this explicitly for the inflaton and the gravitons. Photons are massless: does the same mechanism apply to them? What is the physical reason? And what changes if the universe had a different number of spacetime dimensions?

(b) DBI Inflation. In string theory, inflation can be realized by the motion of a brane in some warped throat (don't worry if you don't understand this, it's not necessary...). In this case, the inflaton field is the position of the brane in the extra dimensions, and its kinetic term is of the DBI form, due to the fact that the brane speed cannot exceed the speed of light. A famous model of this kind of inflation has the action

$$S = -\int \mathrm{d}^4x\sqrt{-g}\left[\frac{\phi^4}{\lambda}\sqrt{1-\lambda g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/\phi^4} - V(\phi)\right]. \tag{6}$$

Show explicitly that inflation can happen on a steep potential, for which ϵ_V and η_V are not small. Calculate the quadratic perturbations: what is the difference from a canonical kinetic term? Since we are not modifying the gravitational action, the spectrum of the tensor perturbations will be the standard one. If we consider the tensor-to-scalar ratio $r = \Delta_T^2/\Delta_{\mathcal{R}}^2$, is this lower or higher than the standard case?

(c) Try to generalize your results to the generic k-inflation Lagrangian:

$$S = \int \mathrm{d}^4x\sqrt{-g}P(\phi, X) \tag{7}$$

where $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$.

(d) Power-law inflation In general, we define inflation as a stage of accelerated expansion of the universe, $\ddot{a} > 0$, in which gravity acts as a repulsive force. To achieve this, we may imagine to have a solution $a \sim t^p$ with $p > 1$. Supposing that the inflaton is a canonical scalar field, find the form of the potential which sources this expansion. In particular, how is p related to the potential parameters? What happens in the limit $p \to infty$? Write down the slow-roll parameters and calculate the number of e-folds as a function of the field range during inflation. Finally, sketch the calculation of the power spectrum of curvature and tensor fluctuations, and determine the parameters, $n_s - 1$, r and n_T.

(e) Consider the equal time 2-point function of a massless scalar in a fixed de Sitter background in real space. You will notice that it diverges at large separation: what is the physical meaning of this infrared divergence?

(f) The de Sitter space is maximally symmetric, which means that it has 10 isometries in 4 space-time dimensions. In inflation, the correlation functions of $\mathcal{R}$ are translationally, rotationally and scale invariant. These are 7 symmetries. What happened to the other 3?

6. Inflationary quantum fluctuations

We are now going to do the calculations of the power spectrum of the inflationary perturbations, starting from simple quantum mechanics to understand their characteristics in an intuitive way. As Coleman said: "the career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction."

(a) Classical oscillator The mathematics of the problem is familiar from classical mechanics. Consider a classical harmonic oscillator of frequency ω. You may want to think of this system as describing the small oscillations of a pendulum, or a mass at the end of a spring, or anything similar. If the frequency is constant in time, the solution is simply given by sines and cosines. Suppose now that the frequency depends on time. You may imagine that the length of the pendulum changes, or that the spring elastic modulus is modified, or whatever. In all of these cases, the equation we are considering is

$$\ddot{y}(t) = -\omega^2(t)y(t) \tag{8}$$

— *Adiabatic approximation* Suppose that the time dependence of ω is *slow*, in the sense that $\dot{\omega}/\omega \ll \omega$. In other words, any change in the system is much slower than the typical time evolution of the unperturbed system. Intuitively, how would you expect the system to behave? Solve approximately the solution to (8) up to first order in $\dot{\omega}/\omega^2$. How does the energy change? Now, construct the action variable $I = \int p\mathrm{d}q$, where the integral is taken over one period. Does it change at lowest order in $\dot{\omega}/\omega^2$? Can you interpret this graphically in phase space?

Notice one interesting thing: if you interpret the variable t in (8) as a spatial variable x, the equation has exactly the form of a time-independent Schroedinger equation in a spatially-varying potential.

— *Sudden approximation* Let us now consider a sudden change in the frequency. The extreme case is a simple jump: for times $t < t_0$, the frequency equals a constant ω_0, and after that it jumps to ω_1, where it stays. Solve eq. (8) in this case, using continuity of the solution at the jump. How does the solution look like? How does the energy change?

(b) Non-relativistic quantum oscillator. Consider now the problem in quantum mechanics, in both the adiabatic and sudden approximations.

Write down the Schroedinger equation and solve for the evolution of the quantum state and energy level, starting from the ground state in the far past. In which state you end up in the far future if the evolution is adiabatic? And what happens in the sudden approximation? Consider both $\omega_0 < \omega_1$ and $\omega_0 > \omega_1$.

Let us now go to the Heisenberg picture. Take the time-independent state $|0\rangle$ equal to the ground state of the oscillator with $\omega = \omega_0$. Construct the time-dependent $\hat{x}$ and $\hat{p}$, and the creation and annihilation operators. Derive the number and energy operators, and look at their expectation values in the $|0\rangle$ state. What values do you get in the far future, in the adiabatic and sudden limits?

(c) A scalar in Minkowski Armed with our mathematics and intuition, the step to quantum field theory is simple. In fact, a free quantum field is just a collection of harmonic oscillators. Take a free real scalar field in Minkowski. Its action is simply

$$S = \int \mathrm{d}^4x \left[\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right] . \tag{9}$$

Write down the equations of motion for the field, and do a spatial Fourier transform. Represent the field as

$$\hat{\phi}(t,\vec{x}) = \int_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} \left[u_k(t)\hat{a}_{\vec{k}} + u_k^*(t)\hat{a}^\dagger_{-\vec{k}}\right] , \tag{10}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operatos, respectively, and $u_k(t)$ satisfies the field equations. Check that the solutions are just plane waves $u_k(t) = \exp(-i\omega t)$ with $\omega^2 = k^2 + m^2$.

Define the conjugate momentum

$$\pi = \frac{\delta S}{\delta \dot{\phi}} , \tag{11}$$

and impose the standard commutation relation at equal time

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\delta_D(\vec{x}-\vec{y}) . \tag{12}$$

Express this in terms of creation and annihilation operators. If we normalize them such that $[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{p}}] = (2\pi)^3\delta_D(\vec{k}-\vec{p})$, how should we normalize the wavefunction?

Now derive the Hamiltonian density via the usual Legendre transform

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} , \tag{13}$$

and express it in terms of the creation and annihilation operators. You will see that it is equal to the number operator familiar from non-relativistic quantum mechanics, plus a zero-point energy that is now divergent as we sum over an infinite number of oscillators. (The only force that cares about this is gravity, and this is in fact the cosmological constant problem. Since in non-gravitational physics we are only concerned with energy differences, we just don't care about this here).

Let's define a state called $|0\rangle$ for lack of fantasy, such that

$$\hat{a}_{\vec{k}}|0\rangle = 0\,. \tag{14}$$

This is the only possible ground state of the system, it has zero particles and minimum energy.

(d) A scalar in de Sitter However, when we have a time-dependent Hamiltonian, things are not that obvious. Let's consider the inflationary case, in which our field lives in (approximately) de Sitter space. Take the flat slicing,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2Ht}\mathrm{d}\vec{x}^2 = \frac{1}{H^2\tau^2}\left(-\mathrm{d}\tau^2 + \mathrm{d}\vec{x}^2\right)\,. \tag{15}$$

The second expression uses conformal time $\mathrm{d}t = a(t)\mathrm{d}\tau$. Traditionally, during inflation we choose $\tau \to -\infty$ at the beginning and $\tau \to 0$ at the end. Check that you get $a(\tau) = -(H\tau)^{-1}$. The action for the field is simply:

$$S = -\int \mathrm{d}^4x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right]\,. \tag{16}$$

Split the field into background and perturbations:

$$\phi(t,\vec{x}) = \phi_0(t) + \varphi(t,\vec{x})\,, \tag{17}$$

and substitute into the action, keeping terms up to second order in φ (so its equations of motion are linear). What happens to the term of first order in the perturbations?

Now focus on the dynamics of perturbations. For simplicity, we will only consider the limit of a massless scalar in a fixed de Sitter background. This means that you will set to zero the potential, and not consider gravitational fluctuations. Write down the field equation for the wavefunction $u_k(t)$, in conformal time. The annoying feature of this equation is that it is not quite a simple harmonic oscillator, because of the damping term with one time derivative. Find a field redefinition that puts the equation in the canonical form $v_k''(\tau) = -\omega^2(\tau)v_k(\tau)$. What is the expression for $\omega^2(\tau)$?

Before solving it, let us check the physics, namely, let us look at $\dot{\omega}/\omega^2$. What is the form of this function? You see that it depends on the parameter $k\tau = \frac{k}{aH}$. This dimensionless parameter is in fact just the ratio of the Hubble radius H^{-1} to the physical wavelength of the perturbations $\lambda = 2\pi a/k$. How does the equation behave when the perturbations are much inside the Hubble radius? And when they are much outside? What happens around Hubble crossing?

We can now solve the equation exactly. Write down the mode functions, which are simple exponentials times a linear function. Then, we need to fix the integration constants. First of all, we have to choose a state. Here things can get ambiguous, because the Hamiltonian depends on time and has no ground state defined at all times. The choice of what we consider a vacuum state fixes one of the integration constants, because if we consider a different linear combination of the two modes we have a different annihilation operator. A very reasonable choice is the so called Bunch-Davies vacuum. Since for $k\tau \to -\infty$ the perturbations have such a small wavelength that they feel they live in Minkowski spacetime, we will choose a state which is the Minkowski vacuum back then. Mathematically, this means that v only has *positive frequency* modes as $k\tau \to -\infty$. The second integration constant is fixed by the canonical normalization of v, v^* which gives $[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{p}}] = (2\pi)^3 \delta_D(\vec{k} - \vec{p})$. Find the correctly normalized solution following these considerations. (Be careful that the condition on v involves a time derivative $\dot{v}(t) = v'(\tau)/a$).

Finally, go back to the ϕ variable, and calculate the two-point function in both real and Fourier space. Calculate the expectation value of the number operator at the end of inflation. How large is it? Why?

Finally, a last small step to get to the sky observations: what is directly related to the CMB perturbations is the variable

$$\mathcal{R} = \frac{H}{\dot{\phi}_0}\varphi\,, \tag{18}$$

which you can interpret as the local perturbation in the scale factor. Compute the power spectrum of $\mathcal{R}$, defined by

$$\langle \mathcal{R}_{\vec{k}} \mathcal{R}_{\vec{p}} \rangle = (2\pi)^3 \delta_D(\vec{k} + \vec{p}) P_{\mathcal{R}}(k)\,. \tag{19}$$

Congratulations! You now understand how tiny, quantum mechanical fluctuations are at the origin of everything. Pretty stunning, isn't it? $\ddot\smile$